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Inria Saclay Centre at Institut Polytechnique de Paris

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ACTIVITY REPORT

Project-Team

TROPICAL

Tropical methods: structures, algorithms and interactions

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP)

DOMAIN

Applied Mathematics, Computation and Simulation

THEME

Optimization and control of dynamic systems

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Project-Team TROPICAL

Creation of the Project-Team: 2018 July 01

Keywords

Computer sciences and digital sciences

A1.2.4. – QoS, performance evaluation

A2.3.3. – Real-time systems

A2.4. – Formal method for verification, reliability, certification

A6.2.5. – Numerical Linear Algebra

A6.2.6. – Optimization

A6.4.2. – Stochastic control

A6.4.6. – Optimal control

A7.2.4. – Mechanized Formalization of Mathematics

A8.1. – Discrete mathematics, combinatorics

A8.2.1. – Operations research

A8.3. – Geometry, Topology

A8.4. – Computer Algebra

A8.9. – Performance evaluation

A8.11. – Game Theory

A9.6. – Decision support

Other research topics and application domains

B4.3. – Renewable energy production

B4.4. – Energy delivery

B4.4.1. – Smart grids

B6.6. – Embedded systems

B8.4. – Security and personal assistance

B8.4.1. – Crisis management

1 Team members, visitors, external collaborators

Research Scientists

- Stephane Gaubert [Team leader, INRIA, Senior Researcher, HDR]
- Marianne Akian [INRIA, Senior Researcher, HDR]
- Xavier Allamigeon [INRIA, Researcher, Corps des Mines, under secondment]
- Matías Bender [INRIA, Researcher, from Feb 2023]
- Yang Qi [INRIA, Starting Research Position, until Oct 2023]
- Cormac Walsh [INRIA, Researcher]

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- Amanda Bigel [INRIA]
- Antoine Béreau [ENS RENNES via Ecole polytechnique]
- Quentin Canu [ENS PARIS-SACLAY, until Aug 2023]
- Pascal Capetillo [INRIA, from Nov 2023]
- Jonathan Hornewall [ENPC, Marne-la-vallée, from Nov 2023]
- Quentin Jacquet [EDF, until Oct 2023]
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- Loic Marchesini [ECOLE POLY PALAISEAU, from Sep 2023]
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Technical Staff

- Benjamin Nguyen-Van-Yen [INRIA, Engineer, from Nov 2023]

Interns and Apprentices

- Loic Marchesini [INRIA, Intern, from Apr 2023 until Sep 2023]

Administrative Assistant

- Hanadi Dib [INRIA]

Visiting Scientists

- Gleb Koshevoi [CEMI, Russian Academy of Sciences, until Jan 2023]
- Gregorio Maljovich [UFRJ, from Jun 2023 until Jul 2023]
- Luz Pascal [Queensland University of Technology, from May 2023 until Jun 2023, PhD student]

2 Overall objectives

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the Tropical project, namely, “structures, algorithms, and interactions”, refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.1 Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [58], combinatorial optimization [64], or automata theory [100]. They also arose in mathematical physics and asymptotic analysis [91, 86]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [79, 115, 105, 84], in algebraic combinatorics [74], and in arithmetics [68]. Also, further applications of tropical methods have appeared, including optimal control [92], program invariant computation [50] and timed systems verification [90], and zero-sum games [1].

The term ‘tropical’ generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations (“log-limits sets” of Bergman, “Maslov dequantization”, or “Viro deformation”). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [115, 96]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [111, 58, 82, 65].

Tropical methods are linked to the fields of positive systems and of metric geometry [98], [12]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert’s projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [88], [2]. Such dynamical systems are of fundamental importance in the study of repeated games [95]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [69]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [6]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [112] or in a PDE setting [60]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [10], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [104, 109].

3 Research program

3.1 Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [71]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [83], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [46], [48], [2].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [72]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the "curse of dimensionality". A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called "max-plus basis methods" [73]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [93, 76]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, "highway hierarchies", etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [51] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [10] and discrete event systems [58, 67, 53]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [2] or metric geometry [11]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3 Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [56],[1], tropical semialgebraic sets [36], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [49], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

3.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \quad , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \quad , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [91], non-archimedean valuations, log-limit sets and Viro's patchworking method [115], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [77, 81].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [52],[7], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [36], [55], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4 Application domains

4.1 Discrete event systems (manufacturing systems, networks, emergency call centers)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [58]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [53].

4.2 Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3 Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [106]. Applications of tropical methods arise in particular from discrete optimization [65], [66], scheduling problems with and-or constraints [97], or product mix auctions [114].

4.4 Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [69], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called "templates" introduced by Sankaranarayanan, Sipma and Manna [108]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [50], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [75, 78] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [6]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5 Social and environmental responsibility

5.1 Impact of research results

The team has developed collaborations on the dimensioning of emergency call centers, with Préfecture de Police (Plate Forme d'Appels d'Urgence - PFAU - 17-18-112, operated jointly by Brigade de sapeurs pompiers de Paris and by Direction de la sécurité de proximité de l'agglomération parisienne) and also with the Emergency medical services of Assistance Publique – Hôpitaux de Paris (Centre 15 of SAMU75, 92, 93 and 94). This work is described further in Section 8.7.1. A recent extension of this work deals with the modelling of medical emergency services, with the project "URGE" which started at the fall 2022, in the framework of the joint INRIA & AP-HP "Bernoulli" lab.

6 Highlights of the year

6.1 Awards

- Stéphane Gaubert obtained the Michel Monpetit – INRIA Prize of Académie des Sciences, 2023.
- Marin Boyet (former PhD of the team) has obtained the PhD prize in Mathematics of Institut polytechnique de Paris, for his work on emergency call centers [63].
- Antoine Béreau obtained the "Distinguished Student Author Award", sponsored by SIGSAM and Maplesoft, at the conference ISSAC (International Symposium on Symbolic and Algebraic Computation), Tromsø, Norway, July 2023.

7 New software, platforms, open data

7.1 New software

7.1.1 Coq-Polyhedra

Name: Coq-Polyhedra

Keywords: Coq, Polyhedra, Automated theorem proving, Linear optimization

Scientific Description: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

Functional Description: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

URL: https://github.com/nhojem/Coq-Polyhedra

Publications: hal-01673390, hal-03151656, hal-03915661, hal-01967575, hal-01967576

Contact: Xavier Allamigeon

Participants: Xavier Allamigeon, Vasileios Charisopoulos, Quentin Canu, Ricardo Katz, Pierre-Yves Strub

Partners: CIFASIS, Ecole Polytechnique

7.1.2 EmergencyEval

Keywords: Dynamic Analysis, Simulation, Ocaml, Emergency, Firefighters, Police

Scientific Description: This software aims at enabling the definition of a Petri network execution semantic, as well as the instanciation and execution of said network using the aforedefined semantic.

The heart of the project dwells in its kernel which operates the step-by-step execution of the network, obeying rules provided by an oracle. This user-defined and separated oracle computes the information necessary to the kernel for building the next state using the current state. The base of our software is the framework for the instanciation and execution of Petri nets, without making assumptions regarding the semantic.

In the context of the study of the dynamics of emergency call centers, a second part of this software is the definition and implementation of the semantic of call centers modelized as Petri nets, and more specifically timed prioritized Petri nets. A module interoperating with the kernel enables to include all the operational specificities of call centers (urgency level, discriminating between operators and callers ...) while guaranteeing the genericity of the kernal which embeds the Petri net formalism as such.

Functional Description: In order to enable the quantitative study of the throughput of calls managed by emergency center calls and the assesment of various organisationnal configurations considered by the stakeholders (firefighters, police, medical emergency service of the 75, 92, 93 and 94 French departments), this software modelizes their behaviours by resorting to extensions of the Petri net formalism. Given a call transfer protocol in a call center, which corresponds to a topology and an execution semantic of a Petri net, the software generates a set of entering calls in accord with the empirically observed statistic ditributions (share of very urgent calls, conversation length), then simulates its management by the operators with respect to the defined protocol. Transitional regimes phenomenons (peak load, support) which are not yet handled by mathematical analysis could therefore be studied. The ouput of the software is a log file which is an execution trace of the simulation featuring extensive information in order to enable the analysis of the data for providing simulation-based insights for decision makers.

The software relies on a Petri net simulation kernel designed to be as modular and adaptable as possible, fit for simulating other Petri-net related phenomenons, even if their semantic differ greatly.

Contact: Xavier Allamigeon

Participants: Xavier Allamigeon, Benjamin Nguyen-van-yen

8 New results

8.1 Optimal control and zero-sum games

8.1.1 Tropical numerical methods for stochastic control problems

Participants: Marianne Akian, Luz Pascal.

In [113], we were interested in the numerical solution of the dynamic programming equation of discrete time stochastic control problems, and we developped and studied several algorithms combining the tropical or the max-plus based numerical method of McEneaney [94, 92], the stochastic max-plus scheme proposed by Zheng Qu [103], and the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [99]. In [37], we also show that in the case of the dynamic programming equation associated to a partially observable Markov Decision Process (POMDP), it is similar to the so called point based algorithms developped in [101, 87, 110], which includes in particular SARSOP algorithm.

In an ongoing work, we are studying the convergence of SARSOP algorithm using the same techniques as in [42].

8.1.2 Highway hierarchies for Hamilton-Jacobi-Bellman (HJB) PDEs

Participants: Marianne Akian, Stéphane Gaubert, Shanqing Liu, Yang Qi.

Hamilton-Jacobi-Bellman equations arise as the dynamic programming equations of deterministic or stochastic optimal control problems. They allow to obtain the global optimum of these problems and to synthetize an optimal feedback control, leading to a solution robust against system perturbations. Several methods have been proposed in the litterature to bypass the obstruction of curse of dimensionality of such equations, assuming a certain structure of the problem, and/or using “unstructured discretizations”, that are not based on given grids. Among them, one may cite tropical numerical method, and probabilistic numerical method. On another direction, “highway hierarchies”, developped by Sanders, Schultes and coworkers [70, 107], initially for applications to on-board GPS systems, are a computational method that allows one to accelerate Dijkstra algorithm for discrete time and state shortest path problems.

The aim of the PhD thesis of Shanqing Liu[89] was to develop new numerical methods to solve Hamilton-Jacobi-Bellman equations that are less sensitive to curse of dimensionality.

In [26], we have developped a multilevel fast-marching method, extending to the PDE case the idea of “highway hierarchies”. Given the problem of finding an optimal trajectory between two given points, the method consists in refining the grid only in a neighborhood of the optimal trajectory, which is itself computed using an approximation of the value function on a coarse grid.

In [18], we have considered general finite horizon deterministic optimal control problems, for which we combine the idea of the multilevel grids around optimal trajectories with the max-plus finite element method solving Hamilton-Jacobi equations introduced in [45].

More recently, we combined these ideas with an adaption of SDDP algorithm to semiconcave problems. We also apply our algorithm to numerically solve the N-body problem.

8.1.3 Relative Krasnoselskii-Mann iteration for irreducible stochastic mean-payoff games and entropy games

Participants: Marianne Akian, Stéphane Gaubert.

We studied in [19] an algorithm solving stochastic mean-payoff games, combining the ideas of relative value iteration and of Krasnoselskii-Mann damping. We analysed its convergence and derived parameterized complexity bounds for several classes of games satisfying irreducibility conditions. This is a joint work with Ulysse Naepels and Basile Terver, two students at École polytechnique, IP Paris.

8.1.4 Escape Rate Games

Participants: Marianne Akian, Stéphane Gaubert, Loic Marchesini.

The aim of the PhD thesis of Loic Marchesini is to study a new class of repeated zero-sum games in which the payoff of one player is the escape rate of a dynamical system whose dynamics depends on the actions of both players. We begin with the case of dynamics obtained from nonexpansive operators. Considering order preserving finite dimensional linear operators over the positive cone endowed with Hilbert's projective (semi-)metric, we recover the matrix multiplication games introduced by Asarin et al. [57], which generalize the joint spectral radius of sets of nonnegative matrices. See also [47]. We established a two-player analogue of "Mañe's Lemma", characterizing the value of the game.

8.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

8.2.1 Volume in Funk geometries

Participants: Cormac Walsh.

This is joint work with Dmitry Faifman (Tel Aviv) and Constantin Vernicos (Montpellier).

We are investigating the volume of metric balls in the Funk geometry [32]. This is an asymmetric metric related to the Hilbert metric; indeed, the Hilbert metric is its symmetrisation. We are interested in it because it is somewhat easier to work with than the Hilbert metric when dealing with volumes, and exhibits similar interesting of behaviour. Given a convex body in $\mathbb{R}^n$ and a radius r, there is a unique point x such that the Funk ball of radius r centered at x minimises the volume over all Funk balls of the same radius. It is natural to conjecture that this minimum volume, which depends on the convex body, it maximised when the body is a Euclidean ball. If this is true, one could recover Blaschke–Santaló inequality by letting the radius tend to zero, and the centro-affine isoperimetric inequality by letting the radius tend to infinity.

Similarly, one could conjecture that, under the assumption that the convex body is centrally symmetric, the minimum volume is minimised when the body is a Hanner polytope. Recall that these are the polytopes that can be constructed from copies of the unit interval by taking products and polar duals. This conjecture interpolates between the Mahler conjecture and Kalai's flag conjecture. We can prove it for unconditional bodies, that is, bodies that are symmetric through reflections in the coordinate hyperplanes.

We have been studying in more detail the volume of balls in the Funk geometry when the convex body is a polytope. Here, as in the case of the Hilbert metric, the volume grows polynomially with order equal to the dimension, and the constant on front of the highest order term depends only on the number of flags of the polytope. Thus, this term does not change when the polytope is perturbed in a way that doesn't change the combinatorics. This motivates us to look at the second highest order term. We have developed the following formula for this in terms of the position of the vertices of the polytope and the vertices of its dual.

Recall that polytopes have the following property: if F and F' are two faces of dimension $i-1$ and $i+1$, respectively, and $F \subset F'$, then there are exactly two faces G of dimension i such that $F \subset G \subset F'$. So, for each $i \in \{0, \dots, n-1\}$ and flag f, there is a unique flag, which we denote $\theta_i f$, that differs from f only by having a different face of dimension i. The group $\langle \theta_0, \dots, \theta_{n-1} \rangle$ generated by these maps is known as the *monodromy group* or *connection group* of the polytope P. To express the second term in the asymptotics of the volume, we will need a particular element of this group, the *complete flip* $\theta := \theta_{n-1} \circ \cdots \circ \theta_0$. With this notation, the second term is then αr^{n-1}, where

$$\alpha = \frac{n}{(n!)^2} \sum_{f \in \mathrm{Flags}(P)} \log\Big(1 - \Big\langle (\theta f)_{n-1}, f_0 \Big\rangle\Big).$$

Here, we are identifying the facet $(\theta f)_{n-1}$ with the associated vertex of the dual polytope.

Thus we get a new centro-affine invariant for polytopes. We have been studying this invariant: we can show that in dimension 2 its only stationary points are the linear images of regular polygons centered at the origin.

8.2.2 Invariant Finsler metrics on symmetric spaces

Participants: Cormac Walsh.

This is joint work with Bas Lemmens (Kent).

We are interested in metrics on symmetric spaces, in particular Finsler metrics that are invariant under the symmetries of the space. In a previous paper [80], it was established that there is such a metric natural associated to every representation of the symmetric space, which depends on the system of weights of the representation. More precisely, one takes the convex hull of the weights and interprets that as the dual ball in the Artin subspace. The lengths of all other vectors can then determined by invariance. We are investigating this correspondence between representations and metrics in more detail. We wish to answer the question, concretely, which known representations correspond to which invariant Finsler metrics?

8.3 Tropical algebra and convex geometry

8.3.1 Formalizing convex polyhedra in Coq

Participants: Xavier Allamigeon, Quentin Canu.

In a joint work with Pierre-Yves Strub (Meta), we have achieved the formal verification of a counterexample of Santos et al. to the so-called Hirsch Conjecture on the diameter of polytopes. In contrast with the pen-and-paper proof, our approach is entirely computational: we implement in Coq and prove correct an algorithm that explicitly computes, within the proof assistant, vertex-edge graphs of polytopes as well as their diameter. The originality of this certificate-based algorithm is to achieve a tradeoff between simplicity and efficiency. Simplicity is crucial in obtaining the proof of correctness of the algorithm. This proof splits into the correctness of an abstract algorithm stated over proof-oriented data types and the correspondence with a low-level implementation over computation-oriented data types. A special effort has been made to reduce the algorithm to a small sequence of elementary operations (e.g., matrix multiplications, basic routines on sets and graphs), in order to make the derivation of the correctness of the low-level implementation more transparent. Efficiency allows us to scale up to polytopes with a challenging combinatorics. For instance, we formally check the two counterexamples of Matschke, Santos and Weibel to the Hirsch conjecture, respectively 20- and 23-dimensional polytopes with 36 425 and 73 224 vertices involving rational coefficients with up to 40 digits in their numerator and denominator.

We also illustrate the performance of the method by computing the list of vertices or the diameter of well-known classes of polytopes, such as (polars of) cyclic polytopes involved in McMullen's Upper Bound Theorem. This work has appeared in the proceedings of the conference CPP'23 [20].

Dealing with polyhedra and their faces raises the problem of formalizing order structures. We study this problem in a joint work [29] with Cyril Cohen (Inria), Kazuhiko Sakaguchi (Inria) and Pierre-Yves Strub (Meta). More precisely, using order structures in a proof assistant naturally raises the problem of working with multiple instances of a same structure over a common type of elements. This goes against the main design pattern of hierarchies used for instance in Coq's MathComp or Lean's mathlib libraries, where types are canonically associated to at most one instance and instances share a common overloaded syntax. We introduce new design patterns to leverage these issues, and apply them to the formalization of order structures in the MathComp library. A common idea in these patterns is underloading, i.e., a disambiguation of operators on a common type. In addition, our design patterns include a way to deal with duality in order structures in a convenient way. We hence formalize a large hierarchy which includes partial orders, semilattices, lattices as well as many variants. We finally pay a special attention to order substructures. We introduce a new kind of structure called prelattice. They are abstractions of semilattices, and allow us to deal with finite lattices and their sublattices within a common signature. As an application, we report on significant simplifications of the formalization of the face lattices of polyhedra in the Coq-Polyhedra library.

8.3.2 Linear algebra over systems

Participants: Marianne Akian, Stéphane Gaubert.

In a joint work with Louis Rowen (Univ. Bar Ilan), [13], we study the properties of "systems". The latter provide a general setting encompassing extensions of the tropical semifields and hyperfields. Moreover, they have the advantage to be well adapted to the study of linear or polynomial equations. In particular, in [13], we characterize the semiring systems which arise from hyperrings.

In [24], we are studying linear algebra properties over a generalization of "systems" called "T-pairs".

8.3.3 Ambitropical convexity and Shapley retracts

Participants: Marianne Akian, Stéphane Gaubert.

Closed tropical convex cones are the most basic examples of modules over the tropical semifield. They coincide with sub-fixed-point sets of Shapley operators – dynamic programming operators of zero-sum games. We study a larger class of cones, which we call "ambitropical" as it includes both tropical cones and their duals. Ambitropical cones can be defined as lattices in the order induced by $\mathbb{R}^n$. Closed ambitropical cones are precisely the fixedpoint sets of Shapley operators. They are characterized by a property of best co-approximation arising from the theory of nonexpansive retracts of normed spaces. Finitely generated ambitropical cones arise when considering Shapley operators of deterministic games with finite action spaces. Moreover, finitely generated ambitropical cones are special polyhedral complexes whose cells are alcoved poyhedra, and locally, they are in bijection with order preserving retracts of the Boolean cube. We also showed that a cone is ambitropical if and only if it is hyperconvex. This is a joint work with Sara Vannucci (Praha). See [27].

8.3.4 Tropical linear regression and applications

Participants: Marianne Akian, Stéphane Gaubert, Yang Qi, Omar Saadi.

In [14], we show that the problem consisting in computing a best approximation of a collection of points by a tropical hyperplane is equivalent to solving a mean payoff game, and also, to compute the maximal radius of an inscribed ball in a tropical polytope. We provide an application to a problem of auction theory – measuring the distance to equilibrium. We also study a dual problem — computing the minimal radius of a circumscribed ball to a tropical polytope – and apply it to the rank-one approximation of tropical matrices and tensors.

8.3.5 Roots over the symmetrized tropical semiring and eigenvalues of tropical symmetric matrices

Participants: Marianne Akian, Stéphane Gaubert.

The tropical semifield can be thought of as the image of a field with a non-archimedean valuation. It allows in this way to study the asymptotics of Puiseux series with complex coefficients. When dealing with Puiseux series with real coefficients and with its associated order, it is convenient to use the symmetrized tropical semiring introduced in [102] (see also [58]), and the signed valuation which associates to any series its valuation together with its sign.

In a work [25] which started during the postdoc of Hanieh Tavakolipour (Amirkabir University of Technology) in the team, we study the roots's multiplicities and the factorization of polynomials over the symmetrized tropical semiring. We then deduce a Descartes' rule of sign over ordered valued fields. This builds in particular on [59] (for multiplicities) and on [36].

In a work with Dariush Kiani and Hanieh Tavakolipour (Amirkabir University of Technology), we study with these tools the asymptotics of eigenvalues and eigenvectors of symmetric positive definite matrices over the field of Puiseux series.

8.3.6 Tropical Systems of Polynomial Equations

Participants: Marianne Akian, Antoine Bereau, Stéphane Gaubert.

The PhD thesis of Antoine Bereau deals with systems of polynomial equations over tropical semifields. In [17, 23], we established a Nullstellenstatz for sparse tropical polynomial systems. We reduce a polynomial system to a linearized system obtained by an appropriate truncation of the Macaulay matrix. Our approach is inspired by a construction of Canny-Emiris (1993), refined by Sturmfels (1994). It leads to an improved estimate of the truncation degree. We also establish a tropical positivstellensatz, allowing one to decide the containment of tropical basic semialgebraic sets. This method leads to the solution of systems of tropical linear equalities and inequalities, which reduces to mean payoff games.

8.3.7 Systems of sparse polynomial equations and convex polytopes

Participants: Matías Bender.

Solving systems of polynomial equations is an intrinsically hard problem. In applications, almost all the systems that we encounter have certain structure, so it is central to take advantage of this to be able to solve bigger inputs. Among the most common structures, we find sparsity: the polynomial are defined by a few set of monomials. If all the polynomial share the same sparsity pattern we say that the system is "unmixed"; otherwise, we call it "mixed". Several strategies had been proposed to deal with sparse polynomial systems, being the more efficients ones the ones to deal with unmixed systems. In our recent paper [30] with Pierre-Jean Spaenlehauer (Inria Nancy), we characterise the conditions under which the specialised strategies developed for unmixed systems can be applied to mixed systems. Our analysis relies on the combinatorics of convex polytopes associated to the input polynomials and toric geometry. Furthermore, we show that deciding whenever this can be done is a NP-hard problem.

8.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

8.4.1 Tropicalization of interior point methods and application to complexity

Participants: Xavier Allamigeon, Stéphane Gaubert, Nicolas Vandame.

It is an open question to determine if the theory of self-concordant barriers can provide an interior point method with strongly polynomial complexity in linear programming. In the special case of the logarithmic barrier, it was shown in [52],[7] that the answer is negative.

In a subsequent work [43] with Abdellah Aznag (Columbia University) and Yassine Hamdi (Ecole Polytechnique), we have studied the tropicalization of the central path associated with the entropic barrier studied by Bubeck and Eldan (Proc. Mach. Learn. Research, 2015), i.e., the logarithmic limit of this central path for a parametric family of linear programs defined over the field of Puiseux series. Our main result is that the tropicalization of the entropic central path is a piecewise linear curve which coincides with the tropicalization of the logarithmic central path studied by Allamigeon et al. in [52],[7].

In the work [40], we have now shown that *none* of the self-concordant barrier interior point methods is strongly polynomial. This result is obtained by establishing that, on parametric families of convex optimization problems, the log-limit of the central path degenerates to the same piecewise linear curve, independently of the choice of the barrier function. We also provided an improved counter example, with an explicit linear program that falls in the same class as the Klee–Minty counterexample, *i.e.*, a n-dimensional combinatorial cube, in which the number of iterations is 2^n.

A key tool in this work consists of metric inequalities, controlling the convergence of the log-images of semialgebraic sets to a polyhedral complex (their tropicalization). Explicit convergence bounds, valid under certain genericity assumptions, have been subsequently established.

In a joint work [39] with Daniel Dadush, Georg Loho, Bento Natura and László Végh, we establish a natural connection between the complexity of interior point methods and that of the simplex method, and deduce combinatorial bounds on the number of iterations. In more details, we introduce a new polynomial-time path-following interior point method where the number of iterations also admits a combinatorial upper bound $O(2^n n^{1.5} \log n)$ for an n-variable linear program in standard form. The number of iterations of our algorithm is at most $O(n^{1.5} \log n)$ times the number of segments of any piecewise linear curve in the wide neighborhood of the central path. In particular, it matches the number of iterations of any path following interior point method up to this polynomial factor. The overall exponential upper bound derives from studying the 'max central path', a piecewise-linear curve with the number of pieces bounded by the total length of $2n$ shadow vertex simplex paths.

8.4.2 Tropical Nash equilibria and complementarity problems

Participants: Xavier Allamigeon, Stéphane Gaubert.

Linear complementarity programming is a generalization of linear programming which encompasses the computation of Nash equilibria for bimatrix games. While the latter problem is PPAD-complete, we show in [44] that the analogue of this problem in tropical algebra can be solved in polynomial time. Moreover, we prove that the Lemke–Howson algorithm carries over the tropical setting and performs a linear number of pivots in the worst case. A consequence of this result is a new class of (classical) bimatrix games for which Nash equilibria computation can be done in polynomial time. This is joint work with Frédéric Meunier (Cermics, ENPC).

8.4.3 Signed Tropicalization of Polars and application to Matrix Cones

Participants: Marianne Akian, Xavier Allamigeon, Stéphane Gaubert.

With Sergey Sergeev (U. Birmingham), we study in [22] the tropical analogue of the notion of polar of a cone over the symmetrized tropical semiring (see for instance [102, 58]). We characterize in particular the tropical polars of sets of nonnegative tropical vectors, and relate them with images by the nonarchimedean valuation of classical polars over real closed nonarchimedean fields. We study in particular cones of matrices, and optimization problems.

8.4.4 Bounds on roots of (multivariate) polynomial systems

Participants: Marianne Akian, Stéphane Gaubert.

With Gregorio Malajovich (Univ. Federal Rio de Janeiro (UFRJ), we study the approximation of the solutions of (classical) polynomial systems, using the roots of some associated tropical polynomial systems, or more generally techniques of tropical geometry. In particular, in a work in progress, we generalize the Cauchy and Lagrange bounds to multivariate polynomials.

8.5 Algebraic aspects of tensors

8.5.1 Algebraic complexity of low-rank approximations

Participants: Yang Qi.

In a joint work with Khazhgali Kozhasov (Université Côte d'Azur), Alan Muniz (Universidade Estadual de Campinas), and Luca Sodomaco (KTH Stockholm) [34], we study the algebraic complexity of Euclidean distance minimization from a generic tensor to the set of rank-one tensors. More precisely, we study the number of complex critical points of this optimization problem, which is called the Euclidean distance (ED) degree of the Segre-Veronese variety in algebraic geometry. We view this invariant as a function of inner products and show this function achieves its minimal value at Frobenius inner product for the matrix case. In addition, we discuss the above optimization problem for other varieties as well.

8.6 Tensor ranks and complexity theory

Participants: Yang Qi.

With Christian Ikenmeyer (Warwick), we study the complexity classes WR which is defined by symmetric rank and its Zariski closure VWR which is defined by symmetric border rank. In particular, we show these two classes equal, i.e., for a given symmetric tensor T with symmetric border rank r, when r is small, the symmetric rank of T is polynomially bounded by r.

8.7 Applications

8.7.1 Performance evaluation of emergency call centers and emergency services

Participants: Xavier Allamigeon, Pascal Capetillo, Stéphane Gaubert, Benjamin Nguyen-Van-Yen.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization (PFAU, "Plate forme d'appels d'urgence") to handle emergency calls to firemen and policemen in the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [53], [54], [61]. In 2019, with four students of École polytechnique, Céline Moucer, Julia Escribe, Skandère Sahli and Alban Zammit, we performed case studies, showing the improvement brought by the two level filtering procedure.

Moreover, in 2019, this work has been extended to encompass the handling of health emergency calls, with a new collaboration, involving responsibles from the four services of medical emergency aid of Assistance Publique – Hôpitaux de Paris (APHP), i.e., with SAMU75, 92, 93, 94, in the framework of a project coordinated by Dr. Christophe Leroy from APHP. As part of his PhD work, Marin Boyet have developed Petri net models capturing the characteristic of the centers (CRRA) handling emergency calls the SAMU, in order to make dimensioning recommendations. Following this, we have been strongly solicited by APHP during the pandemic of Covid-19 in order to determine crisis dimensioning of SAMU.

In parallel, we have further investigated the theoretical properties of timed Petri nets with preselection and priority routing. We represent the behavior of these systems by piecewise affine dynamical systems. We use tools from the theory of nonexpansive mappings to analyze these systems. We establish an equivalence theorem between priority-free fluid timed Petri nets and semi-Markov decision processes, from which we derive the convergence to a periodic regime and the polynomial-time computability of the throughput. More generally, we develop an approach inspired by tropical geometry, characterizing the congestion phases as the cells of a polyhedral complex. These results are illustrated by the application to the performance evaluation of emergency call centers of SAMU in the Paris area. These results have been published in [35].

In [38], we provided explicit formulæ allowing one to compute the time needed by a call center to return to a stationary state after a bulk of calls. This is based on a turnpike-type theorem for Markov decision processes.

These results are also presented in the Phd thesis [63].

In a followup work, in the framework of the URGE project (see Section 10.3.3 below) we are extending these models in order to evaluate the dimensioning of emergency departments. This is the object of the PhD work of Pascal Capetillo (which started in Nov. 2023).

8.7.2 Optimal pricing of energy contracts

Participants: Stéphane Gaubert, Quentin Jacquet.

The PhD thesis of Quentin Jacquet [85] was cosupervised by Stéphane Gaubert, Clémence Alasseur (EDF Labs), and Wim van Ackooij (EDF Labs). It concerns the application of bilevel programming methods to the pricing of electricity contracts. We investigated in [16] a new model of customer's response, based on a quadratic regularization. We showed that this model has qualitative properties and a realism similar to the classical models based on the logit-response, while being amenable to mathematical programming and polyhedral techniques, and so to exact solutions, via a reduction to quadratic complementary problems. An application to a set of instances representative of French electricity contracts was also developed in [16].

In [41], we developed a model representing the response of a population of customers to dynamic price offers. This lead to a mean-field Markov decision model, with ergodic cost. We showed in particular the optimality of pricing policies characterized by periodic discounts.

The preprint [28] develops a mean field game model to model an incentive pricing scheme, in which agents compete to get the best reward.

The conference article [21] develops a method to compute an optimal menu of contracts of a prescribed cardinality, optimizing the income of a provider, taking into account the agent's characteristics.

9 Bilateral contracts and grants with industry

9.1 Bilateral contracts with industry

Participants: Stéphane Gaubert.

- Optimal pricing of energy and services. Collaboration of Stéphane Gaubert with Clémence Alasseur and Wim Van Ackooij, from EDF Labs, with the Phd Work of Quentin Jacquet (CIFRE PhD, 2020-2023), supervised by Stéphane Gaubert.

10 Partnerships and cooperations

10.1 International initiatives

10.1.1 Participation in other International Programs

Bilateral project FACCTS with the University of Chicago

Participants: Stéphane Gaubert.

Title: Tropical geometry of deep learning

Partner Institution(s): University of Chicago (Statistics) and Ecole polytechnique.

Program: Bilateral project FACCTS, between the University of Chicago (Statistics) – Lek-Heng Lim– and Ecole polytechnique – Stéphane Gaubert–

Partners: Lek-Heng Lim

Date/Duration: 2021-2023

10.2 International research visitors

10.2.1 Visits of international scientists

Other international visits to the team

Gleb Koshevoy

Status Senior Researcher.

Institution of origin: Institute of Information Transmission Problems (Kharkevich Institute), Russian Academy of Sciences.

Country: Russia.

Dates: November 1, 2022 to January 31, 2023.

Context of the visit: INRIA Saclay invited professor programme. Gleb Koshevoy is invited to collaborate with the Tropical team, on topics at the interface of algebraic combinatorics, discrete convexity and tropical geometry. The collaboration program covers the study of ambitropical and hyperconvexity, their interactions with F-polynomials, and tropical positivstellensatze.

Mobility program/type of mobility: research stay.

Luz Pascal

Status PhD.

Institution of origin: Queensland University of Technology.

Country: Australia.

Dates: May 30, 2023 to June 20, 2023.

Context of the visit: Participation of Marianne Akian to the supervision of Luz Pascal on the algorithms for Partially Observable Markov Decision Processes and their applications to adaptive management of species.

Mobility program/type of mobility: research stay.

Gregorio Malajovich

Status Senior Researcher.

Institution of origin: Mathematical Institute, Universidade Federal do Rio de Janeiro.

Country: Brazil.

Dates: June 22, 2023 to July 7, 2023.

Context of the visit: Collaboration with the TROPICAL team (Marianne Akian and Stéphane Gaubert) on tropical geometry applied to the localization of roots of polynomial systems.

Mobility program/type of mobility: research stay.

10.2.2 Visits to international teams

Research stays abroad

Matías Bender

Visited institution: University of Texas at San Antonio.

Country: US

Dates: October 30, 2023 to November 3, 2023.

Context of the visit: Collaboration with Alperen Ergür.

Mobility program/type of mobility: research stay.

10.3 National initiatives

10.3.1 ANR

Participants: Marianne Akian, Xavier Allamigeon, Cormac Walsh, Stéphane Gaubert.

- Project ANR HilbertXField ("Géométries de Hilbert sur tout corps valué"). ANR leader: Antonin Guilloux. Partners: IMJ-PRG/OURAGAN (Sorbonne Université, pole leader Antonin Guilloux), CMAP/TROPICAL (Inria, pole leader: Cormac Walsh), Institut Fourier (Grenoble, pole leader Anne Parreau).

10.3.2 Programme Gaspard Monge pour l'optimisation, la recherche opérationnelle et leurs interactions avec les sciences des données

Participants: Matías Bender.

- SOAP - Sparsity in Optimization via Algebra and Polynomials, with Elias Tsigaridas (IMJ), project funded by FMJH within the PGMO programme.

10.3.3 Joint INRIA & AP-HP Bernoulli lab project: "URGE"

- The project URGE (Analyse des parcours patients aux URgences et optimisation des prises en charGE), started at the fall 2022, in the framework of the joint INRIA & AP-HP Bernoulli lab. The goal of the project is to develop modelling, simulation, performance analysis, and visualization tools, in order to help physicians to optimize the staffing of emergency services. This collaborative project, of four years, involves the Tropical and Aviz teams from INRIA Saclay, the Dyogene team from INRIA Paris, and the Fédération Hospitalo-Universitaire (FHU) / IMPEC Improving Emergency Care, AP-HP / Sorbonne Université / INSERM. The project is led by X. Allamigeon (Tropical) and Y. Yordanov (AP-HP, Saint-Antoine) and involves S. Gaubert, B. Nguyen (Tropical), Ch. Fricker (Dyogene), J.D. Fekete (Aviz).

11 Dissemination

11.1 Promoting scientific activities

11.1.1 Scientific events: organisation

General chair, scientific chair

- Stéphane Gaubert is the coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorhip program, operated by Fondation Mathématique Jacques Hadamard, supported by EDF, see Pgmo site.

Member of the organizing committees

- Stéphane Gaubert co-organizes the "Séminaire Parisien d'Optimisation" at Institut Henri Poincaré.

Organization of special sessions at conferences

- Marianne Akian, organization of minisymposium "Équations d'Hamilton-Jacobi-Bellman et de jeux à champ moyen" at *SMAI 2023*, May 22-26, 2023, Le Gosier en Guadeloupe.

- Marianne Akian, organization of minisymposium "Computational Tropical Geometry" at *SIAM Conference on Applied Algebraic Geometry (AG23)*, Eindhoven, The Netherlands.

- Matìas Bender, organization of minisymposium "Efficient Symbolic and Numerical Algorithms for Polynomial Systems" at *SIAM Conference on Applied Algebraic Geometry (AG23)*, Eindhoven, The Netherlands.

- Matìas Bender, organization of invited session "Nonlinear Algebra and its Application" at *PGMO Days 2023*, Palaiseau, France.

- Yang Qi, organization of minisymposium "Geometry in optimization and numerical (multi)linear algebra" at *SIAM Conference on Applied Algebraic Geometry (AG23)*, Eindhoven, The Netherlands.

11.1.2 Scientific events: selection

Chair of conference program committees

- Stéphane Gaubert, co-chair of the PGMO Days 2023, EDF Labs, 28-29 Nov.

Member of the conference program committees

- Marianne Akian, member of the Program Commitee of ISSAC'2023.
- Marianne Akian, member of the Scientific Commitee of 11th French Biennal of SMAI (SMAI 2023).
- Stéphane Gaubert, member of the scientific committee of the Mathematics and Decision Conference, 11 – 15 December 2023 Place: UM6P Ben Guerir, Morocco.
- Stéphane Gaubert, member of the scientific committee of the workshop Modélisation des Systèmes Réactifs (MSR'23), Toulouse.

11.1.3 Journal

Member of the editorial boards

- Stéphane Gaubert: member of the editorial board of Journal of Dynamics and Games, Linear and Multilinear Algebra, RAIRO, Springer-SMAI book series.

11.1.4 Invited talks

- Marianne Akian:
  - Minisymposium "Manifestations of the Mixed Volume" at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk : "Cauchy and Lagrange Bounds for Multivariate Polynomials" (work with Stéphane Gaubert and Gregorio Malajovich).
  - Minisymposium "Numerical methods for Hamilton-Jacobi equations and their applications" at 10th International Congress on Industrial and Applied Mathematics (ICIAM 2023), August 2023, Tokyo, Japan. Talk: "Tropical and multi-level numerical methods for solving optimal control problems" (work with Stéphane Gaubert and Shanqing Liu).
  - Invited talk at Mathematics & Decision Conference, 11-15 December 2023, UM6P Ben Guerir, Morocco. Title: "Repeated zero-sum games: from operators to algorithms and tropical geometry".
- Xavier Allamigeon
  - Journée des lauréats des prix de l'Académie des Sciences, section "Mécanique et Informatique", Paris, France, mai 2023. Titre : *A Formal Disproof of the Hirsch Conjecture.*
  - Exposé invité à la conférence "Highlights of Algorithms", Prague, République Tchèque, juin 2023. Titre : *No self-concordant barrier interior point method is strongly polynomial.*
  - Minisymposium "Computational Tropical Geometry" at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk: "Tropical Nash Equilibria and Oriented Matroids" (work with Stéphane Gaubert and Frédéric Meunier).
  - Exposé invité au 27ème séminaire "Polyèdres et Optimisation Combinatoire", Paris, France, décembre 2023. Titre : *No self-concordant barrier interior point method is strongly polynomial.*
- Matìas Bender
  - Minisymposium "Manifestations of the Mixed Volume" at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk: "Dimension Results for Polynomial Systems Over Complete Toric Varieties".

  - Minisymposium “Recent Advances on Polynomial System Solving” at 10th International Congress on Industrial and Applied Mathematics (ICIAM 2023), August 2023, Tokyo, Japan. Talk: “Dimension Results for Polynomial Systems Over Complete Toric Varieties”.
  - Weekly seminar of the FCFS at *Universidad Autónoma de Chiapas*.

- Antoine Béreau:
  - Minisymposium “Manifestations of the Mixed Volume” at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk: “The Tropical Nullstellensatz and Positivstellensatz for Sparse Polynomial Systems” (work with Marianne Akian and Stéphane Gaubert).

- Stéphane Gaubert:
  - 25th Conference of the International Linear Algebra Society, Madrid, June 2023 (Invited Minisymposium). Talk: “Turing-model complexity estimates in geometric programming and application to the computation of the spectral radius of nonnegative tensors.”.
  - Jon-Shmuel Halfway to Twelfty celebration, ENPC, July 2023. Talk: “Ambitropical convexity, injective hulls of metric spaces, and mean-payoff games.”
  - Minisymposium “Computational Tropical Geometry” at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk: “Ambitropical Convexity, Injective Hulls and Mean-Payoff Games” (work with Marianne Akian and Sara Vannucci).
  - Minisymposium “Advances in Optimization” at 10th International Congress on Industrial and Applied Mathematics (ICIAM 2023), August 2023, Tokyo, Japan. Talk: “Tropical convexity: application to linear programming and mean-payoff games”.
  - Combinatorics and Arithmetic for Physics, IHES, Nov. 2023. Talk: “The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems, and Parametric Mean-Payoff Games” (joint work with Marianne Akian and Antoine Béreau).

- Yang Qi:
  - Optimization Seminar, Telecom Paris, April 2023, Talk: “Tropical linear regression and low-rank approximation — a first step in tropical data analysis”.
  - MC2 Seminar, ENS Lyon, May 2023, Talk: “Tropical linear regression and low-rank approximation — a first step in tropical data analysis”.
  - Frontiers of computational mathematics, Kunming, China, August 2023, Talk: “Tropical data analysis”.

- Nicolas Vandame:
  - Minisymposium “Computational Tropical Geometry” at SIAM Conference on Applied Algebraic Geometry (AG23), July 2023, Eindhoven, The Netherlands. Talk: “No Self-Concordant Barrier Interior Point Method Is Strongly Polynomial” (work with Stéphane Gaubert and Xavier Allamigeon).

- Cormac Walsh:
  - “Workshop on Ordered Vector Spaces and Positive Operators”, Wuppertal, Germany, 29 March–1 April 2023. Talk: “Order and metric structures on cones”.

11.1.5 Research administration

Inria research administration

- Marianne Akian: Elected member of Inria’s Scientific Board.
- Matìas Bender: Member of the scientific committee of INRIA Saclay.

Other research administration

- Marianne Akian: “Secrétaire Générale” of SMAI since July 2023.

11.2 Teaching - Supervision - Juries

11.2.1 Teaching

- Marianne Akian
  - Course “Markov decision processes: dynamic programming and applications” joint between (3rd year of) ENSTA and M2 “Mathématiques et Applications”, U. Paris Saclay, “Optimization”, 30 hours.
- Xavier Allamigeon
  - Petites classes et encadrement d’enseignements d’approfondissement de Recherche Opérationnelle en troisième année à l’École Polytechnique (programme d’approfondissement de Mathématiques Appliquées) (niveau M1).
  - Cours “Theoretical Aspects of Linear Programming” du M2 “Optimisation” de l’Université Paris Saclay.
  - Co-responsabilité du programme d’approfondissement en mathématiques appliquées (troisième année) à l’École Polytechnique (jusqu’à septembre 2023).
- Matìas Bender
  - Exercises classes for the second year of Bachelor program of Ecole polytechnique in the framework of a “Chargé d’ensegnement”.
  - Course “Computational commutative algebra” at the CIMPA summer school *Algebraic and Tropical Methods for Solving Differential Equations*, Oaxaca, Mexico.
- Antoine Béreau
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Amanda Bigel
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Stéphane Gaubert
  - Co-head of the Master “Optimization” of University Paris-Saclay and IPP.
  - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
  - Course “Algèbre tropicale pour le contrôle optimal et les jeux” of “Contrôle, Optimisation et Calcul des Variations” (COCV) of M2 “Mathématiques et Applications” of Sorbonne University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [62].
- Shanqing Liu
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Nicolas Vandame
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.

11.2.2 Supervision

- Matìas Bender directed the M2 thesis of Tom Baumbach at the Technische Universität Berlin.
- PhD: Quentin Jacquet, registered at IPP (EDMH) since November 2020, thesis supervisor, Stéphane Gaubert, co-supervised by Clémence Alasseur and Wim van Ackooij. The defense took place on October 24, 2023.
- PhD: Shanqing Liu, registered at IPP (EDMH) since September 2020, thesis supervisor, Marianne Akian, co-supervised by Stéphane Gaubert. The defense took place on December 22, 2023.
- PhD in progress: Quentin Canu, registered at Univ. Paris Saclay since October 2020, thesis supervisor: Georges Gonthier (INRIA), cosupervision: Xavier Allamigeon and Pierre-Yves Strub (LIX).
- PhD in progress: Antoine Bereau, registered at IPP (EDMH) since September 2021. Thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD in progress: Nicolas Vandame, registered at IPP (EDMH) since September 2021, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Amanda Bigel, registered at IPP (EDMH) since September 2022, main thesis supervisors: Cormac Walsh et Constantin Vernicos, thesis supervisor: Stéphane Gaubert.
- PhD in progress: Loic Marchesini, registered at IPP (EDMH), since September 2023. Thesis supervisor Marianne Akian, co-supervised by Stéphane Gaubert.
- PhD in progress: Pascal Capetillo, registered at IPP (EDMH), since November 2023. Thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Jonathan Hornewall, registered at Marne-la-vallée, ENPC (maths and STIC), since November 2023. Thesis supervisor: Vincent Leclere, cosupervision: Stéphane Gaubert.

11.2.3 Juries

- Marianne Akian
  - Jury (and reviewer) of the HDR of Vincent Leclere, Ecole des Ponts ParisTech, April 17, 2023.
  - Jury of the PhD thesis of Christina Katsamaki, Sorbonne Univerity, June 21, 2023.
  - Jury of the PhD thesis of Manh Hung LE, Université de Limoges, Dec. 20, 2023.
  - Jury of the PhD thesis of Shanqing Liu, Ecole polytechnique, Dec. 22, 2023.
- Stéphane Gaubert
  - Member of the HCERES visiting committee of CEREMADE, October 2023.
  - Member of “Comité de Sélection” (MdC, Optimization) at Nice, Spring 2023.
  - Jury of the PhD thesis of Quentin Jacquet, Ecole polytechnique, October 24, 2023.
  - Jury the PhD thesis of Francisco Venegas, University of Chile, July 2023.

11.2.4 Communications at conferences and seminars

- Marianne Akian
  - Minisymposium “Équations d’Hamilton-Jacobi-Bellman et de jeux à champ moyen” à SMAI 2023, May 22-26, 2023, Le Gosier en Guadeloupe. Exposé: “A multi-level fast-marching method for the minimum time problem” (travail avec Stéphane Gaubert et Shanqing Liu).
  - 13th ISDG Workshop, June 7-9, 2023, Naples, Italy, Session “Numerics and Theory for Differential Games, in Memory of Maurizio Falcone”, Talk : “Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration”.

  - 17th International Conference on Reachability Problems (RP 2023). Talk : "Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration".

- Antoine Béreau:
  - Journées nationales de calcul formel, Marseille, March 2023. Talk: "The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems".
  - Rencontres Doctorales Lebesgue 2023, Nantes, April 2023. Exposé: "Un tour d'horizon des mathématiques tropicales".
  - International Symposium on Symbolic and Algebraic Computation 2023, July 24-27, 2023. Talk: "The Tropical Nullstenllensatz and Positivstellensatz for Sparse Polynomial Systems" (work with Marianne Akian and Stéphane Gaubert).
- Stéphane Gaubert:
  - 48TH International symposium on mathematical foundations of computer science (MFCS 2023). Talk : "Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration".
  - Seminar of of the Dyogene INRIA Team, "Nonarchimedean convexity and applications", April 2023.
  - Seminar of the verification group at IRIF, "Entropy games", May 2023.
- Shanqing Liu:
  - IFAC World Congress 2023, Yokohama, Japan. Talk: "An Adaptive Multi-Level Max-Plus Method for Deterministic Optimal Control Problems".
  - PGMO days, Palaiseau, November 2023. Talk: "Semiconcave Dual Dynamic Programming and Its Application to N-body Problems" (work with Marianne Akian, Stéphane Gaubert and Yang Qi).
- Cormac Walsh:
  - "Positivity XI", Ljubljana, Slovenia, 10–14 July 2023. Talk: "Hilbert geometries isometric to Banach spaces".

12 Scientific production

12.1 Major publications

[1] M. Akian, S. Gaubert and A. Guterman. 'Tropical polyhedra are equivalent to mean payoff games'. In: *Internat. J. Algebra Comput.* 22.1 (2012), pp. 1250001, 43. DOI: 10.1142/S0218196711006674. eprint: 0912.2462. URL: http://dx.doi.org/10.1142/S0218196711006674.

[2] M. Akian, S. Gaubert and R. Nussbaum. 'Uniqueness of the fixed point of nonexpansive semidifferentiable maps'. In: *Transactions of the American Mathematical Society* 368.2 (Feb. 2016). Also arXiv:1201.1536. DOI: 10.1090/S0002-9947-2015-06413-7. URL: https://hal.inria.fr/hal-00783682.

[3] M. Akian, S. Gaubert and R. Bapat. 'Non-archimedean valuations of eigenvalues of matrix polynomials'. In: *Linear Algebra and its Applications* 498 (June 2016). Also arXiv:1601.00438, pp. 592–627. DOI: 10.1016/j.laa.2016.02.036. URL: https://hal.inria.fr/hal-01251803.

[4] M. Akian, S. Gaubert and C. Walsh. 'The max-plus Martin boundary'. In: *Doc. Math.* 14 (2009), pp. 195–240.

[5] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'Combinatorial simplex algorithms can solve mean payoff games'. In: *SIAM J. Opt.* 24.4 (2015), pp. 2096–2117. eprint: 1309.5925.

[6] X. Allamigeon, S. Gaubert, E. Goubault, S. Putot and N. Stott. 'A scalable algebraic method to infer quadratic invariants of switched systems'. In: *Proceedings of the International Conference on Embedded Software (EMSOFT)*. Best paper award. The extended version of this conference article appeared in *ACM Trans. Embed. Comput. Syst., 15(4):69:1–69:20, September 2016*. 2015.

[7] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'What Tropical Geometry Tells Us about the Complexity of Linear Programming'. In: *SIAM Review* 63.1 (4th Feb. 2021), pp. 123–164. DOI: 10.1137/20M1380211. URL: https://hal.inria.fr/hal-03505719.

[8] J. Bolte, S. Gaubert and G. Vigeral. 'Definable zero-sum stochastic games'. In: *Mathematics of Operations Research* 40.1 (2014). Also arxiv1301.1967, pp. 171–191. DOI: 10.1287/moor.2014.0666. URL: http://dx.doi.org/10.1287/moor.2014.0666.

[9] S. Friedland, S. Gaubert and L. Han. 'Perron–Frobenius theorem for nonnegative multilinear forms and extensions'. In: *Linear Algebra and its Applications* 438.2 (2013), pp. 738–749. DOI: 10.1016/j.laa.2011.02.042. URL: https://hal.inria.fr/hal-00782755.

[10] S. Gaubert and T. Lepoutre. 'Discrete limit and monotonicity properties of the Floquet eigenvalue in an age structured cell division cycle model'. In: *J. Math. Biol.* (2015). DOI: 10.1007/s00285-015-0874-3. eprint: 1301.2151. URL: http://dx.doi.org/10.1007/s00285-015-0874-3.

[11] S. Gaubert and G. Vigeral. 'A maximin characterization of the escape rate of nonexpansive mappings in metrically convex spaces'. In: *Math. Proc. of Cambridge Phil. Soc.* 152 (2012). https://arxiv.org/abs/1012.4765, pp. 341–363. URL: http://dx.doi.org/10.1017/S0305004111000673.

[12] C. Walsh. 'The horofunction boundary and isometry group of the Hilbert geometry'. In: *Handbook of Hilbert Geometry*. Vol. 22. IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2014. URL: https://hal.inria.fr/hal-00782827.

12.2 Publications of the year

International journals

[13] M. Akian, S. Gaubert and L. Rowen. 'Semiring systems arising from hyperrings'. In: *Journal of Pure and Applied Algebra* 228.6 (June 2024), p. 107584. DOI: 10.1016/j.jpaa.2023.107584. URL: https://inria.hal.science/hal-03792658.

[14] M. Akian, S. Gaubert, Y. Qi and O. Saadi. 'Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria'. In: *SIAM Journal on Discrete Mathematics* 37.2 (24th May 2023), pp. 632–674. DOI: 10.1137/21M1428297. URL: https://inria.hal.science/hal-03504875.

[15] X. Allamigeon, S. Gaubert and F. Meunier. 'Tropical Complementarity Problems and Nash Equilibria'. In: *SIAM Journal on Discrete Mathematics* 37.3 (3rd Aug. 2023), pp. 1645–1665. DOI: 10.1137/21M1446861. URL: https://hal.science/hal-04355016.

[16] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Quadratic regularization of bilevel pricing problems and application to electricity retail markets'. In: *European Journal of Operational Research* 313.3 (Mar. 2024), pp. 841–857. DOI: 10.1016/j.ejor.2023.05.006. URL: https://inria.hal.science/hal-03504874.

International peer-reviewed conferences

[17] M. Akian, A. Béreau and S. Gaubert. 'The Tropical Nullstellensatz and Positivstellensatz for Sparse Polynomial Systems'. In: *ISSAC '23: Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation*. ISSAC 2023 - International Symposium on Symbolic and Algebraic Computation. Tromsø, Norway: ACM, 2023. DOI: 10.1145/3597066.3597089. URL: https://inria.hal.science/hal-04117544.

[18] M. Akian, S. Gaubert and S. Liu. 'An Adaptive Multi-Level Max-Plus Method for Deterministic Optimal Control Problems'. In: *Proceedings of the 22nd IFAC World Congress*. IFAC World Congress 2023. Yokohama, Japan, 9th July 2023. URL: https://inria.hal.science/hal-04406978.

[19] M. Akian, S. Gaubert, U. Naepels and B. Terver. 'Solving Irreducible Stochastic Mean-Payoff Games and Entropy Games by Relative Krasnoselskii-Mann Iteration'. In: MFCS 2023 - 48th International Symposium on Mathematical Foundations of Computer Science. Vol. 272. Bordeaux, France: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 28th Aug. 2023. DOI: 10.4230/LIPIcs.MFCS.2023.10. URL: https://inria.hal.science/hal-04365602.

[20] X. Allamigeon, Q. Canu and P.-Y. Strub. 'A Formal Disproof of Hirsch Conjecture'. In: *Proceedings of the 12th ACM SIGPLAN International Conference on Certified Programs and Proofs*. CPP 2023 - 12th ACM SIGPLAN International Conference on Certified Programs and Proofs. Boston, United States: ACM, 11th Jan. 2023, pp. 17–29. DOI: 10.1145/3573105.3575678. URL: https://inria.hal.science/hal-04368266.

[21] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'A Quantization Procedure for Nonlinear Pricing with an Application to Electricity Markets'. In: *Proceedings of IEEE CDC 2023*. 62nd IEEE Conference on Decision and Control. Singapore (SG), Singapore, 13th Dec. 2023. URL: https://hal.science/hal-04052232.

Reports & preprints

[22] M. Akian, X. Allamigeon, S. Gaubert and S. Sergeev. *Signed tropicalization of polar cones*. 28th Dec. 2023. URL: https://inria.hal.science/hal-04365674.

[23] M. Akian, A. Béreau and S. Gaubert. *The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems*. 10th Dec. 2023. URL: https://inria.hal.science/hal-04333931.

[24] M. Akian, S. Gaubert and L. Rowen. *Linear algebra over T-pairs*. 8th Oct. 2023. URL: https://inria.hal.science/hal-04406848.

[25] M. Akian, S. Gaubert and H. Tavakolipour. *Factorization of polynomials over the symmetrized tropical semiring and Descartes' rule of sign over ordered valued fields*. 13th Jan. 2023. URL: https://inria.hal.science/hal-03941832.

[26] M. Akian, S. Gaubert and S. Liu. *A Multi-Level Fast-Marching Method For The Minimum Time Problem*. 19th Mar. 2023. URL: https://inria.hal.science/hal-04365665.

[27] M. Akian, S. Gaubert and S. Vannucci. *Ambitropical geometry, hyperconvexity and zero-sum games*. 6th July 2023. URL: https://inria.hal.science/hal-03504873.

[28] C. Alasseur, E. Bayraktar, R. Dumitrescu and Q. Jacquet. *A Rank-Based Reward between a Principal and a Field of Agents: Application to Energy Savings*. 31st July 2023. URL: https://hal.science/hal-03770115.

[29] X. Allamigeon, Q. Canu, C. Cohen, K. Sakaguchi and P.-Y. Strub. *Design patterns of hierarchies for order structures*. 28th Feb. 2023. URL: https://inria.hal.science/hal-04008820.

[30] M. R. Bender and P.-J. Spaenlehauer. *Dimension results for extremal-generic polynomial systems over complete toric varieties*. 12th May 2023. URL: https://inria.hal.science/hal-04102564.

[31] P. Cannarsa, S. Gaubert, C. Mendico and M. Quincampoix. *Analysis of the vanishing discount limit for optimal control problems in continuous and discrete time*. 12th June 2023. URL: https://inria.hal.science/hal-04365643.

[32] D. Faifman, C. Vernicos and C. Walsh. *Volume growth of Funk geometry and the flags of polytopes*. 15th June 2023. URL: https://hal.science/hal-04403794.

[33] S. Friedland and S. Gaubert. *Bit-complexity estimates in geometric programming, and application to the polynomial-time computation of the spectral radius of nonnegative tensors*. 25th Jan. 2023. URL: https://inria.hal.science/hal-04365626.

[34] K. Kozhasov, A. Muniz, Y. Qi and L. Sodomaco. *On the minimal algebraic complexity of the rank-one approximation problem for general inner products*. 26th Sept. 2023. URL: https://inria.hal.science/hal-04408246.

12.3 Cited publications

[35] X. Allamigeon, M. Boyet and S. Gaubert. 'Piecewise Affine Dynamical Models of Timed Petri Nets – Application to Emergency Call Centers'. In: *Fundamenta Informaticae* 183.3-4 (2021), pp. 169–201. URL: https://hal.archives-ouvertes.fr/hal-02550006.

[36] X. Allamigeon, S. Gaubert and M. Skomra. 'Tropical spectrahedra'. In: *Discrete and Computational Geometry* 63 (Feb. 2020), pp. 507–548. DOI: 10.1007/s00454-020-00176-1. URL: https://hal.inria.fr/hal-01422639.

[37] M. Akian, J.-P. Chancelier, L. Pascal and B. Tran. 'Tropical numerical methods for solving stochastic control problems'. In: MTNS 2022 - 25th International Symposium on Mathematical Theory of Networks and Systems. Bayreuth (DE), Germany, 12th Sept. 2022. URL: https://hal.inria.fr/hal-03944216.

[38] X. Allamigeon, M. Boyet and S. Gaubert. 'Computing Transience Bounds of Emergency Call Centers: a Hierarchical Timed Petri Net Approach'. In: PETRI NETS 2022: Application and Theory of Petri Nets and Concurrency. Vol. 13288. PETRI NETS 2022: Application and Theory of Petri Nets and Concurrency, Springer Lecture Notes in Computer Sciences. Bergen, Norway: Springer, 2022, pp. 90–112. DOI: 10.1007/978-3-031-06653-5_5. URL: https://hal.inria.fr/hal-03913405.

[39] X. Allamigeon, D. Dadush, G. Loho, B. Natura and L. Vegh. 'Interior point methods are not worse than Simplex'. In: 2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS). Denver, United States: IEEE, 31st Oct. 2022, pp. 267–277. DOI: 10.1109/FOCS54457.2022.00032. URL: https://hal.inria.fr/hal-03915650.

[40] X. Allamigeon, S. Gaubert and N. Vandame. 'No self-concordant barrier interior point method is strongly polynomial'. In: STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing. Rome Italy, France: ACM, 20th June 2022, pp. 515–528. DOI: 10.1145/3519935.3519997. URL: https://hal.inria.fr/hal-03915670.

[41] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Ergodic control of a heterogeneous population and application to electricity pricing'. In: IEEE CDC 2022. Cancun, Mexico, 6th Dec. 2022. URL: https://hal.science/hal-03629189.

[42] M. Akian, J.-P. Chancelier and B. Tran. *Tropical Dynamic Programming for Lipschitz Multistage Stochastic Programming*. 13th Dec. 2020. URL: https://hal.inria.fr/hal-03059701.

[43] X. Allamigeon, A. Aznag, S. Gaubert and Y. Hamdi. *The tropicalization of the entropic barrier*. 2020. URL: https://hal.archives-ouvertes.fr/hal-03151661.

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[45] M. Akian, S. Gaubert and A. Lakhoua. 'The max-plus finite element method for solving deterministic optimal control problems: basic properties and convergence analysis'. In: *SIAM J. Control Optim.* 47.2 (2008), pp. 817–848. DOI: 10.1137/060655286. URL: http://www.arxiv.org/abs/math.OC/0603619.

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