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Project-Team VEGAS

Effective Geometric Algorithms for Visibility and Surfaces

Nancy - Grand Est

Theme : Algorithms, Certification, and Cryptography

Activity Report

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2. Overall Objectives

2.1. Overall Objectives

VEGAS is a research project of LORIA (Lorraine Research Laboratory in Computer Science and Applications), a laboratory shared by INRIA (National Institute for Research in Computer Science and Control), CNRS (National Center for Scientific Research), Université Henri Poincaré Nancy 1, Université Nancy 2, and INPL (National Engineering Institute of Lorraine).

The main scientific objective of the VEGAS research team is to *contribute to the development of an effective geometric computing* dedicated to *non-trivial geometric objects*. Included among its main tasks are the study and development of new algorithms for the manipulation of geometric objects, the experimentation of algorithms, the production of high-quality software, and the application of such algorithms and implementations to research domains that deal with a large amount of geometric data, notably solid modeling and computer graphics.

Computational geometry has traditionally treated linear objects like line segments and polygons in the plane, and point sets and polytopes in three-dimensional space, occasionally (and more recently) venturing into the world of non-linear curves such as circles and ellipses. The methodological experience and the know-how accumulated over the last thirty years have been enormous.

For many applications, particularly in the fields of computer graphics and solid modeling, it is necessary to manipulate more general objects such as curves and surfaces given in either implicit or parametric form. Typically such objects are handled by approximating them by simple objects such as triangles. This approach is extremely important and it has been used in almost all of the usable software existing in industry today. It does, however, have some disadvantages. Using a tessellated form in place of its exact geometry may introduce spurious numerical errors (the famous gap between the wing and the body of the aircraft), not to mention that thousands if not hundreds of thousands of triangles could be needed to adequately represent the object. Moreover, the curved objects that we consider are not necessarily everyday three-dimensional objects, but also abstract mathematical objects that are not linear, that may live in high-dimensional space, and whose geometry we do not control. For example, the set of lines in 3D (at the core of visibility issues) that are tangent to three polyhedra span a piecewise ruled quadratic surface and the lines tangent to a sphere correspond, in projective five-dimensional space, to the intersection of two quadratic hypersurfaces.

Effectiveness is a key word of our research project. By requiring our algorithms to be effective, we imply that the algorithms should be *robust, efficient,* and *versatile*. By robust we mean algorithms that do not crash on degenerate inputs and always output topologically consistent data. By efficient we mean algorithms that run reasonably quickly on realistic data where performance is ascertained both experimentally and theoretically. Finally, by versatile we mean algorithms that work for classes of objects that are general enough to cover realistic situations and that account for the *exact geometry* of the objects, in particular when they are curved.

3. Scientific Foundations

3.1. Theory and applications of three-dimensional visibility

The notion of 3D visibility plays a fundamental role in computer graphics. In this field, the determination of objects visible from a given point, the extraction of shadows or of penumbra boundaries are examples of visibility computations. In global illumination methods, (e.g. radiosity algorithms), it is necessary to determine, in a very repetitive manner, if two points of a scene are mutually visible. The computations can be excessively expensive. For instance, in radiosity, it is not unusual that 50 to 70% of the simulation time is spent answering visibility queries.

Objects that are far apart may have very complicated and unintuitive visual interactions, and because of this, visibility queries are intrinsically global. This partially explains that, until now, researchers have primarily used ad hoc structures, of limited scope, to answer specific queries on-the-fly. Unfortunately, experience has shown that these structures do not scale up. The lack of a well-defined mathematical foundation and the non-exploitation of the intrinsic properties of 3D visibility result in structures that are not usable on models consisting of many hundreds of thousands of primitives, both from the viewpoint of complexity and robustness (geometric degeneracies, aligned surfaces, etc.).

We have chosen a different approach which consists of computing ahead of time (that is, off-line) a 3D global visibility structure for which queries can be answered very efficiently on-the-fly (on line). The 3D visibility complex – essentially a partition of ray space according to visibility – is such a structure, recently introduced in computational geometry and graphics [58], [64]. We approach 3D global visibility problems from two directions: we study, on the one hand, the theoretical foundations and, on the other hand, we work on the practical aspects related to the development of efficient and robust visibility algorithms.

From a theoretical point of view, we study, for example, the problem of computing lines tangent to four among k polytopes. We have shown much better bounds on the number of these tangents than were previously known [3]. These results give a measure of the complexity of the vertices (cells of dimension 0) of the visibility complex of faceted objects, in particular, for triangulated scenes.

From a practical point of view, we have, for example, studied the problem of the complexity for these 3D global visibility structures, considered by many to be prohibitive. The size of these structures in the worst case is $O(n^4)$, where n is the number of objects in the scene. But we have, in fact, shown that when the objects are uniformly distributed, the complexity is linear in the size of the input [6]. This probabilistic result does not prejudice the complexity observed in real scenes where the objects are not uniformly distributed. However, initial empirical studies show that, even for real scenes, the observed complexity is largely inferior to the theoretical worst-case complexity, as our probabilistic result appears to indicate.

We are currently working on translating these positive signs into efficient algorithms. We are studying new algorithms for the construction of the visibility complex, putting the accent on the complexity and the robustness.

3.2. Reliable geometric computations on curves and surfaces

Simple algebraic surfaces cover a variety of forms sufficient for representing the majority of objects encountered in the fields of design, architecture and industrial manufacturing. For instance, it has been estimated that 95% of all mechanical pieces can be well modeled by quadric patches (degree 2 surfaces, including planes, spheres, cylinders and cones) and torii [65]. It is important, then, to be able to process these surfaces in a robust and efficient manner.

In comparison with polygonal representations, modeling and manipulating scenes made of curved objects pose a large variety of new issues and require entirely different tools. It is for instance no longer realistic to assume that simple operations like intersecting two primitives take constant time. The usual notion of complexity has to be revised and needs to incorporate the arithmetic complexity of operations.

Geometric computing with curved objects is plagued with robustness issues. The numerical instability of geometric algorithms is intimately linked to the double nature of geometric objects. Indeed, a geometric object is two things: a combinatorial structure which encodes the incidence properties between the elements constituting the object and numerical quantities (coordinates, equations) describing the embedding of the object in space. Manipulating geometric data, without breaking the consistency constraints that govern the relation between combinatorial and numerical quantities, is usually hard and has led to the unfolding of the *exact geometric computing* paradigm.

The dependence of combinatorial decisions on numerical computations is encapsulated in the notion of *geometric predicates*. When working with algebraic objects, evaluating a geometric predicate often means determining the sign of a polynomial expression in the coefficients of the input. This sign encodes the answer to simple geometric queries like “are three given points aligned?” or “is a given line tangent to a given surface?”. The paradigm of exact geometric computing requires the predicates to be evaluated exactly, ensuring that the branching of the algorithm are correct, that the software will not crash, loop indefinitely or output a wrong answer, and thus that the topological structure of the output is correct.

In the context of exact geometric computing, we work on key problems involving curved objects, mainly two-dimensional curves, and low-degree three-dimensional surfaces such as quadrics. For instance, we study intersections of quadrics both from an algorithmic and an algebraic-geometric point of view. On the algorithmic side, we work on finding simple and usable parameterizations of the intersection of two arbitrary quadrics. On the algebraic side, we deal with finding simple (and ideally optimal) geometric predicates for classifying the intersection pattern and the positional relationship of two quadrics.

We also work on computing arrangements of curved objects, i.e. the partitioning of space induced by the objects, such as arrangements of curves on a surface, or arrangements of quadrics in 3D space. Note that intersections of 2 and 3 quadrics are building blocks for the constructions of quadric arrangements. We work on constructing simpler sub-arrangements, like the BRep (Boundary Representation) of a solid model (CSG). Exact CSG-to-BRep conversion is a key and long-standing problem in CAGD, where many conventional modelers work with volumes and rendering software based on the global illumination approach need surface patches.

Finally, we deal with geometric problems where low-degree surfaces appear indirectly, not in the input but as intermediate structures. A major problem in this category is the computation of the Voronoi diagram, or medial axis, of polyhedra in 3D. In particular, we work on the simpler instance where only lines and line segments in 3D are considered, the bisectors of pairs of lines being quadric surfaces.

4. Application Domains

4.1. Computer graphics

Our main application domain is photorealistic rendering in computer graphics. We are especially interested in the application of our work to virtual prototyping, which refers to the many steps required for the creation of a realistic virtual representation from a CAD/CAM model.

When designing an automobile, detailed physical mockups of the interior are built to study the design and evaluate human factors and ergonomic issues. These hand-made prototypes are costly, time consuming, and difficult to modify. To shorten the design cycle and improve interactivity and reliability, realistic rendering and immersive virtual reality provide an effective alternative. A virtual prototype can replace a physical mockup for the analysis of such design aspects as visibility of instruments and mirrors, reachability and accessibility, and aesthetics and appeal.

Virtual prototyping encompasses most of our work on effective geometric computing. In particular, our work on 3D visibility should have fruitful applications in this domain. As already explained, meshing objects of the scene along the main discontinuities of the visibility function can have a dramatic impact on the realism of the simulations.

4.2. Solid modeling

Solid modeling, i.e., the computer representation and manipulation of 3D shapes, has historically developed somewhat in parallel to computational geometry. Both communities are concerned with geometric algorithms and deal with many of the same issues. But while the computational geometry community has been mathematically inclined and essentially concerned with linear objects, solid modeling has traditionally had closer ties to industry and has been more concerned with curved surfaces.

Clearly, there is considerable potential for interaction between the two fields. Standing somewhere in the middle, our project has a lot to offer. Among the geometric questions related to solid modeling that are of interest to us, let us mention: the description of geometric shapes, the representation of solids, the conversion between different representations, data structures for graphical rendering of models and robustness of geometric computations.

4.3. Fast prototyping

We work in collaboration with CIRTES on rapid prototyping. CIRTES, a company based in Saint-Dié-des-Vosges, has designed a technique called Stratoconception® where a prototype of a 3D computer model is constructed by first decomposing the model into layers and then manufacturing separately each layer, typically out of wood of standard thickness (e.g. 1 cm), with a three-axis CNC (Computer Numerical Controls) milling machine. The layers are then assembled together to form the object. The Stratoconception® technique is cheap and allows fast prototyping of large models.

When the model is complex, for example an art sculpture, some parts of the models may be inaccessible to the milling machine. These inaccessible regions are sanded out by hand in a post-processing phase. This phase is very consuming in time and resources. We work on minimizing the amount of work to be done in this last phase by improving the algorithmic techniques for decomposing the model into layers, that is, finding a direction of slicing and a position of the first layer [62].

5. Software

5.1. QI: Quadrics Intersection

Participants: Laurent Dupont, Sylvain Lazard, Sylvain Petitjean.

QI stands for "Quadrics Intersection". QI is the first exact, robust, efficient and usable implementation of an algorithm for parameterizing the intersection of two arbitrary quadrics, given in implicit form, with integer coefficients. This implementation is based on the parameterization method described in [10], [55], [56], [57] and represents the first complete and robust solution to what is perhaps the most basic problem of solid modeling by implicit curved surfaces.

QI is written in C++ and builds upon the LiDIA computational number theory library [39] bundled with the GMP multi-precision integer arithmetic [38]. QI can routinely compute parameterizations of quadrics having coefficients with up to 50 digits in less than 100 milliseconds on an average PC; see [10] for detailed benchmarks.

Our implementation consists of roughly 18,000 lines of source code. QI has being registered at the Agence pour la Protection des Programmes (APP). It is distributed under the free for non-commercial use INRIA license and will be distributed under the QPL license in the next release. The implementation can also be queried via a web interface [40].

Since its official first release in June 2004, QI has been downloaded six times a month on average and it has been included in the geometric library EXACUS developed at the Max-Planck-Institut für Informatik (Saarbrücken, Germany). QI is also used in a broad range of applications; for instance, it is used in photochemistry for studying the interactions between potential energy surfaces, in computer vision for computing the image of conics seen by a catadioptric camera with a paraboloidal mirror, and in mathematics for computing flows of hypersurfaces of revolution based on constant-volume average curvature.

5.2. Isotop: Topology and Geometry of Planar Algebraic Curves

Participants: Jinsan Cheng, Sylvain Lazard, Luis Peñaranda, Marc Pouget.

ISOTOP is a Maple software for computing the topology of an algebraic plane curve, that is, for computing an arrangement of polylines isotopic to the input curve. This problem is a necessary key step for computing arrangements of algebraic curves and has also applications for curve plotting. This software has been developed since 2007 in collaboration with F. Rouillier from INRIA Paris - Rocquencourt (SALSA). It is based on the method described in [20] which incorporates several improvements over previous methods. In particular, our approach does not require generic position (nor shearing) and avoids the computations of sub-resultant sequences. Our preliminary implementation is competitive with other implementations (such as ALCIX and INSULATE developed at MPII Saarbrücken, Germany and TOP developed at Santander Univ., Spain). It performs similarly for small-degree curves and performs significantly better for higher degrees, in particular when the curves are not in generic position.

5.3. CGAL: Computational Geometry Algorithms Library

Participants: Sylvain Lazard, Luis Peñaranda, Marc Pouget.

Born as a European project, CGAL (http://www.cgal.org) has become the standard library for computational geometry. It offers easy access to efficient and reliable geometric algorithms in the form of a C++ library. CGAL is used in various areas needing geometric computation, such as: computer graphics, scientific visualization, computer aided design and modeling, geographic information systems, molecular biology, medical imaging, robotics and motion planning, mesh generation, numerical methods...

M. Pouget is co-author and maintainer, with F. Cazals from INRIA Sophia Antipolis - Méditerranée (ABS team), of two packages released in the version (3.3) of the library. These packages belong to the geometry processing part, they enable the *Approximation of Ridges and Umbilics* [45] and the *Estimation of Local Differential Properties* [46] on triangulated surface meshes.

In computational geometry, many problems lead to standard, though difficult, algebraic questions such as computing the real roots of a system of equations, computing the sign of a polynomial at the roots of a system, or determining the dimension of a set of solutions. we want to make state-of-the-art algebraic software more accessible to the computational geometry community, in particular, through the computational geometric library CGAL. On this line, S. Lazard and L. Peñaranda proposed an extension to the already existing Number Types package. It consists in adding a multiple-precision floating-point arithmetic, and the corresponding interval arithmetic; these number types are based on the libraries MPFR and MPFI. They also developed a model of the *Univariate Algebraic Kernel* concept for algebraic computations. This package improves, for instance, the efficiency of the computation of arrangements of polynomial functions in CGAL [28]. This implementation uses the RS library developed by F. Rouillier at INRIA Paris - Rocquencourt (SALSA) for isolating real roots of polynomials. All these packages have been reviewed and accepted or tentatively accepted by the editorial board of CGAL and should be released next year.

6. New Results

6.1. Effective 3D global visibility, theory and applications

Participants: Guillaume Batog, Hazel Everett, Xavier Goaoc, Sylvain Lazard, Sylvain Petitjean, Linqiao Zhang.

In recent years, our activity in the area of 3D visibility focused on three main directions: (i) the computation and complexity analysis of the *3D visibility complex,* (ii) the computation and complexity analysis of the boundary of shadows cast by area light sources, and (iii) the study of some fundamental questions in geometric transversal theory.

The 3D visibility complex is a partition of the space of rays according to visibility. The questions we consider are two-fold. On one hand, we study its size, both experimentally and theoretically. This size is reflected in the number of unobstructed line segments tangent to four objects of the scene; while this number can be $\Theta(n^4)$ in unstructured scenes, its behavior for more *realistic* input is still not understood. On the other hand, we work on the computation of the *3D Visibility skeleton,* a substructure of the visibility complex. A few years ago we presented a worst-case near-optimal sweep-plane algorithm for computing this structure in a restrictive setting [3]. Over the last years, we have been working on an efficient and reliable implementation of that algorithm, and on the development of an algorithm in a more general setting.

Shadows play a central role in human perception and a wide variety of approaches have been considered for simulating and rendering them. Unfortunately, computing realistic shadows efficiently is a very difficult problem, particularly in the case of non-point light sources, due to the complicated internal structure that such shadows may have. The only current solution for that problem goes through a discretization of area light sources by many point light sources; shadows cannot be certified with this approach and their structure is lost. As a first approximation, one can compute the main shadow boundaries, that is the boundary between the regions in full-light, penumbra, and umbra. Specifically, a point is in the umbra if it does not see any part of any light source; it is in full light if it sees entirely all the light sources; otherwise, it is in the penumbra. These boundaries are traced out by one-parameter families of segments that belong to the 3D Visibility skeleton. On one hand, we try to estimate the complexity of these shadow boundaries, both in theory and in practice. On the other hand, we work on applying our methods for computing the visibility skeleton to this particular problem.

The study of fundamental properties of sets of lines, that is, *line geometry,* underlies many algorithmic questions including 3D visibility computations. A few years ago, we generalized Helly's classical theorem to sets of line transversals to disjoint unit balls in $\mathbb{R}^d$ [5]. This theorem builds on two types of results: a local Helly-type theorem, also called a *pinning theorem,* and a bound on the topological complexity of the space of line transversals, obtained by counting so-called *geometric permutations*. Our activity focuses on extending both types of results.

6.1.1. *Visibility complex and skeleton*

This year, our PhD student Linqiao Zhang, co-supervised with S. Whitesides at McGill University, defended her Ph.D. thesis entitled *On the three-dimensional visibility skeleton: implementation and analysis* [11]. She developed a first implementation of our sweep-plane algorithm that computes robustly the vertices of the visibility skeleton of a set of convex polyhedra in generic position. This implementation is a key element of a prototype we developed for computing limits of umbra [24] (see below); it also allowed us to study experimentally the size of the 3D visibility skeleton in a random setting and, in particular, to show that the constant involved in the asymptotic complexity is small [11] [66]. Related to this work, we studied various predicates, arising in three-dimensional visibility, concerning line transversals to lines and segments in 3D. In particular, we computed the degrees of standard methods of evaluating these predicates. We showed that the degrees of some of these methods are surprisingly high (up to 168), which may explain why computing line transversals with finite-precision floating-point arithmetic is prone to error [11]; this work also published this year in the journal *Computational Geometry, Theory and Applications* [16].

We also obtained a new result on the complexity of the set free lines and free lines segments tangent to balls in 3D. In particular, we proved that, in the presence of n disjoint unit balls, the set of free line segment, that is the so-called visibility complex, has complexity $\Theta(n^4)$. This result is very surprising because it was natural to conjecture that the visibility complex of fat objects of similar size had a lower worst-case complexity than that for thin triangles. Our result settles negatively this conjecture, and shows exactly the opposite, that is, that fatness and similarity, alone, do not reduce the worst-case complexity of that structure. This result was submitted to the *Symposium on Computational Geometry* in December 2009 [35].

In the context of the PhD thesis of Guillaume Batog, we are working on an invariant-based method for designing efficient evaluation strategies for predicates. In a nutshell, the idea is to find a group action on the input space of a predicate with two properties: it has few orbits, and any two inputs in the same orbit are equivalent for the predicate. The invariants of that action are known, under certain conditions, to distinguish the various orbits, and can then be used to decide the predicate. This year, we unfolded this approach for predicates on line systems that arise in visibility computations, for instance deciding the number of lines intersecting four given lines. A communication is in preparation.

6.1.2. Shadow computations

We published this year two results on the umbra and penumbra cast by non-trivial light sources.

First, we published the first non-trivial bounds on the size of the umbra region. These result show that the umbra can be surprisingly complicated, even in the presence of disjoint fat obstacles; for instance, we proved that the umbra cast on a plane by a segment light source in the presence of two fat polytopes of size n, can have up to $\Theta(n)$ connected components in the worst case. These results were published in the journal *Computational Geometry, Theory and Applications* [13].

We also developed, over the last years, a new method for the exact computation of shadow boundaries. The associated prototype software is, up to our knowledge, the first implementation that computes exactly such shadow boundaries of polygonal light sources in the presence of polyhedral obstacles for nontrivial scenes (*e.g.,* 50 polytopes with 1 500 vertices). Using this implementation, we observed experimentally a significant gap between the size of the visibility skeleton and the, much smaller, complexity of the shadow boundaries. We presented these results at the *European Workshop on Computational Geometry* [24].

6.1.3. Pinning and line transversals

In the last few years, we strived to understand the combinatorial properties of so-called *isolated line transversals,* also called *pinnings*: these lines meet every member in a family of objects, but they lose this property when subject to any arbitrarily small perturbation. We obtained a *basis theorem* when the objects are disjoint balls in $\mathbb{R}^d$ in 2006 [5]: any pinning of a line by disjoint balls in $\mathbb{R}^d$ contains a pinning of size at most $2d-1$. We strengthened this result in three ways. First, we proved that this constant, $2d-1$, is best possible for all dimension, a result obtained in 2008. In 2009, we presented this result at the *Eurocomb* conference [21] and submitted a complete version for journal publication. Then, we extended this result to pinnings of a line by possibly intersecting smooth convex sets [37]. Last, in 2008 we extended this result to pinnings by polyhedra in $\mathbb{R}^3$. This year, we added to this extension a complete description of minimal pinning configurations; these results were submitted to the *Symposium on Computational Geometry* in December 2009 [29].

An order in which a line can intersect a collection of geometric objects is called a *geometric permutation* of that collection. When the objects are disjoint balls, the geometric permutations count the connected components of the space of line transversals ; more generally, it gives a lower bound on its topological complexity. Substantial effort has been devoted to understanding how the geometry of the objects constrains the number of geometric permutations; the asymptotic behavior of the maximum number of geometric permutations of n disjoint convex sets in $\mathbb{R}^3$ has remained an open problem for two decades. We have been working on a new approach to this question. The idea is to analyze how the knowledge of a bound for constant-size problems (say, knowing that the maximum number of geometric permutations for 10 objects is at most 24) helps bounding the asymptotic behavior for n objects. This year, we made a first step in this direction: we delineated the main growth rates (constant, polynomial, exponential) through a generalization of Sauer's Lemma for *shatter functions* of set systems.

6.2. Certified geometric computing for curves and surfaces

Participants: Jinsan Cheng, Laurent Dupont, Hazel Everett, Sylvain Lazard, Luis Peñaranda, Maria Pentcheva, Sylvain Petitjean, Marc Pouget, George Tzoumas.

6.2.1. Voronoi diagram of polyhedra in 3D

We are working on the problem of computing the medial axis or Voronoi diagram of polyhedra in 3D. These structures are largely used in applications; the medial axis is, for instance, a way of representing a shape by its topological skeleton. Such a diagram is a partition of space into cells, each of which consists of the points closest to one particular object than to any other. Moreover, the set of points equidistant to two lines (or to a line and a point) is a quadric and the set of points equidistant to three lines is the intersection of two quadrics. While such structures are well-understood in the plane and for simple situations in higher dimensions (e.g. for sets of points), a lot remains to be done; for example, there is no working solution for computing exactly the medial axis of a polyhedron.

We started a few years ago by considering the Voronoi diagram of lines and we finally published this year, in the journal *Discrete and Computational Geometry,* some very nice results characterizing the topology of the Voronoi diagrams of three lines [15]. We proved, in particular, that the topology is invariant for lines in general position and we obtained a monotonicity property on the arcs of the diagram which has important algorithmic implications. The proof technique, which relies heavily upon modern tools of computer algebra, is also of great interest in its own right.

We have worked during the last two years on the problem of extending these results to the case of three lines in arbitrary positions providing the first complete characterization of the Voronoi diagram of any three lines. Preliminary results were presented this year at the European Workshop on Computational Geometry [25]. These results also yielded a new algorithm, fundamental for handling robustness issues, for sorting points along the arcs of Voronoi diagrams of lines (with rational coefficients) using only rational linear tests.

6.2.2. Adjacency graph of an arrangement of integer quadrics

We have presented, a couple years ago, a complete, exact and efficient algorithm and its implementation for computing the adjacency graph of an arrangement of quadrics with integer coefficients [54]. This year, we completed this work and submitted it to the *Journal of Symbolic Computation* [34]. This algorithm builds upon our previous work on parameterization of intersections of quadrics. Intersecting a parameterized intersection of two quadrics with a third quadric leads to finding the real zeros of polynomials of degree at most 8 with possibly algebraic coefficients. Experiments show that our implementation outperforms past approaches when dealing with generic situations, even in case of large bitsize and/or algebraic coefficients. This efficiency, even over algebraic extensions and with large bit-size numbers, is due, during the computation of the roots of univariate polynomials, to the use of the bitstream Descartes algorithm, which replaces each number by a series of certified approximations. In non-generic situations, the current implementation is hampered by slow gcd computations over algebraic extensions.

6.2.3. Algebraic tools for geometric computing

In computational geometry, many problems lead to standard, though difficult, algebraic questions such as computing the real roots of a system of equations, computing the sign of a polynomial at the roots of a system, or determining the dimension of a set of solutions. Our goal is two-fold. First, we want to make state-of-the-art algebraic software more accessible to the computational geometry community, in particular, through the computational geometric library CGAL. Second, our goal is to demonstrate to which extent such state-of-the-art certified algebraic root-finding systems can be used in geometric algorithms to obtain certified constructions involving curved objects without hindering performance. We have presented some results in these two directions at the 8th International Symposium on Experimental Algorithms, SEA'09 [22], and we submitted this year, to the CGAL editorial board, a package which is a model of the *Univariate Algebraic Kernel* concept for algebraic computations (see section Software).

6.2.4. Topology and geometry of algebraic curves: Isotop

We worked over the last years on the problem of computing, in a certified way, the topology of algebraic curves, that is, computing an arrangement of polylines isotopic to the input curve. The objective here is to compute efficiently and in a certified way arrangements of algebraic curves. A necessary key step is to compute the topology of any given curve. Moreover, geometric information, such as the position of singular and critical points, is also mandatory for computing arrangements of several curves using a sweep-line algorithm. A difficulty is to compute efficiently this information for the given coordinate system even if the curve is not in generic position; previous practical approaches shear back and forth the coordinate system, which is time consuming. In addition, costly computations with polynomials whose coefficients are algebraic should be avoided. We have recently presented an algorithm that incorporates several improvements over previous methods and overcome these difficulties. In particular, our approach does not require generic position nor shearing. This work has been presented this year at the 25th annual Symposium on Computational Geometry [20], and was submitted to the journal of *Mathematics in Computer Science* [31]. We have also developed a Maple implementation of this algorithm which is very promising (see section Software).

6.2.5. Constant-complexity geometric problems and algebraic invariants

We have continued revisiting some key constant-complexity geometric problems with a view to better understand their degenerate instances and the geometric predicates underlying their detection. For that, we mostly rely on classical tools, in particular (classical) algebraic invariant century, which was perceived as a bridge between geometry and algebra by the mathematicians of the 19th century (culminating with Klein's Erlangen Program, and the view of geometry as the study of the properties of a space invariant under the action of a group of transformations). Last year, we studied the relative position of two plane projective conics and showed that it can be characterized by predicates of bidegree at most $(6, 6)$ in the coefficients of the input conics [63], improving upon previous results. By relative position we mean the morphology of the intersection, the rigid isotopy class and which conic is inside the other when applicable. Analyzing the algebraic invariant theory of pencils of conics, we constructed a special conic – called a combinant – invariantly attached to a given pencil and showed how the projective type of this combinant, encoded by its inertia, is characteristic of the intersection type of the two conics in most cases. However, the problem was treated purely algebraically and the results have no obvious geometric meaning: why such inertia is characteristic of such intersection pattern is obscure. This year, we reproved essentially the same results, but using an entirely new approach which has the benefit of making perfectly clear the geometric meaning of the inertia of the combinant conic and overall bringing substantial geometric insight to the problem [23]. The key intermediate tool we use that illuminates this interpretation is the Bezoutian. We are working on extending these results to other primitives.

6.2.6. Bounded-curvature path planning

We studied the problem of computing shortest paths of bounded curvature that visits a sequence of n points in order. This problem, which has been open for about 15 years, is crucial for path planning of car-like robots in the presence of polygonal obstacles. We proved that, under some conditions, this problem reduces to optimizing a convex function over a convex n-dimensional domain. This result reveals a fundamental property of curvature-constrained paths among polygonal obstacles, and it provides the first efficient solution for this long-standing problem. This result has been submitted to the *Symposium on Computational Geometry* in December 2009 [36].

6.2.7. Embedding geometric structures

This year, we started work on the problem of embedding geometric objects on a grid of $\mathbb{R}^3$. Essentially all industrial applications take, as input, models defined with a fixed-precision floating-point arithmetic, typically doubles. As a consequence, geometric objects constructed using exact arithmetic must be embedded on a fixed-precision grid before they can be used as input in other software. More precisely, the problem is, given a geometric object, to find a similar object representable with fixed-precision floating-point arithmetic, where similar means topologically equivalent, close according to some distance function, etc. We started working on the problem of rounding polyhedral subdivisions on a grid of $\mathbb{R}^3$, where the only known method, due to Fortune, induces a blow-up in the complexity that is inacceptable in practice. We worked so far on the simpler problem of embedding convex polyhedra. We also showed negative results that, even in the plane, convex polygons cannot be rounded while conserving both convexity and proximity of the rounded vertices. This project is joint work with Mark de Berg (Eindhoven University), Dan Halperin (Tel Aviv University) and Olivier Devillers (Geometrica, INRIA).

6.3. Other results

We initiated a collaboration with O. Devillers (INRIA, Sophia-Antipolis) and D. Attali (Gipsa, Grenoble) on *smoothed complexity* analysis of geometric data structures. In many cases, the worst-case complexity analysis poorly represents the practical performances of algorithms or data structures. The smoothed complexity, which aims at supplementing this gap, is defined as the maximum over the inputs of the expected complexity over small perturbations of that input. We obtained some preliminary results on the smoothed number of extreme points of a convex point set subject to random noise; we submitted these results to the *Symposium on Computational Geometry* in December [30], [26].

We completed some research on farthest-site Voronoi diagrams of polygonal sites of total complexity n. We proved that the combinatorial complexity of such diagrams is $O(n)$, and we presented an $O(n \log^3 n)$ time algorithm to compute it. These results were submitted to the journal *Transactions on Algorithms* [33].

One of our earlier results on computational topology was accepted this year in the journal *Computational Geometry, Theory and Applications* [12]. We considered the Fréchet distance between two curves in the plane is the minimum length of a leash that allows a dog and its owner to walk along their respective curves, from one end to the other, without backtracking. We proposed a natural extension of Fréchet distance to more general metric spaces, which requires the leash itself to move continuously over time. For example, for curves in the punctured plane, the leash cannot pass through or jump over the obstacles.

Finally, two of our earlier results in graph drawing were also accepted or published this year in the journal of *Discrete and Computational Geometry* [17], [18]. The first result shows, in particular, that any planar graph with n vertices can be point-set embedded with at most one bend per edge on a given set of n points in the plane. An implication of this result is that any number of planar graphs admit a simultaneous embedding without mapping with at most one bend per edge.

6.4. International initiatives

6.4.1. Associated Teams and Other International Projects

- McGill-VEGAS associated team. This INRIA program is a joint project between our group and the computational geometry laboratory of McGill University (Montréal), and in particular Sue Whitesides. This associated team was started in 2002 under the name McGill-ISA before the creation of VEGAS. The research theme is 3D visibility [1], [2], [3], [6], [9], [11], [13], [14], [16], [24], [35], [41], [42], [44], [47], [51], [52], [53], [59], [60], [61], [66] and, more generally, computational geometry. In this context, we organize regular international workshops (1st to 8th McGill - INRIA Workshop on Computational Geometry in Computer Graphics, 2002 - 2009) which regroup, for one week, 15 to 25 researchers from around the world. Many research projects were initiated during these workshops on the theme of 3D visibility and line geometry [2], [3], [9], [13], [41], [43], [44], [48], [49], [52], [53]. Note finally that our former Ph.D. student, L. Zhang, whom defended in August, was co-supervised with S. Whitesides. Also, S. Whitesides has moved in August to the university of Victoria (Canada).

  In the context of this cooperation, INRIA supported VEGAS up to 5 Keuros, and the Canadian side provided an equivalent support through S. Whitesides' personal NSERC grant.

- KAIST-INRIA associated team. This INRIA program is a joint project between VEGAS and the Theory of Computation Laboratory of the KAIST University of Daejeon, in Korea, more particularly the group of Otfried Cheong. It started in 2008, following a 2-years PHC grant. The research theme is Discrete and Computational Geometry, in general, with a particular emphasis on questions where both continuous and discrete aspects come into play and interact. We organized a kick-off workshop in 2008, and continued the collaboration in 2009 through mutual visits, including a two-months visit to LORIA by Hyo-Sil Kim a PhD student from KAIST. The projects on which we collaborate include line geometry [21], [27], [32], [29], bounded curvature path planning [36] and geometric data structures [33], [50].

  In 2009, this cooperation was supported for 13 kE by INRIA and for 2.4 kE by our partners.

- Sylvain Petitjean started a collaboration with Pr. Gert Vegter of the University of Groningen on "Certified Geometric Approximation". This collaboration is funded by the Netherlands Organization for Scientific Research (NWO) - 2008–2012. Fatma Senguler Ciftci started her PhD co-advised by S. Petitjean and G. Vegter in June 2009.

6.5. Visiting scientists

International visitors:

- Hyo-Sil Kim, KAIST, May–June, 2 months;
- Mark de Berg, TU Eindhoven, Jul., 3 days;
- Dan Halperin, Tel Aviv University, Jul., 3 days;
- Esther Ezra, Tel-Aviv University, Dec., 1 week;
- Michael Hemmer, MPI, 1 week over the year.

Visitors from France:

- Cyril Nicaud, Univ. de Marne-la-Vallée, 3x2 days;
- Olivier Devillers, INRIA, Sophia, 2x2 days;
- Jean Ponce, ENS Paris, March, 2 days;
- Fabrice Rouillier, INRIA Rocquencourt, July, 1 week.

International visits:

- Xavier Goaoc, New York Univ., USA (2 weeks), Univ. Konstanz, Germany (1 week), KAIST, Korea (1 week), Univ. Calgary, Canada (1 week);
- Sylvain Lazard, Univ. of Victoria, Canada (1 week), McGill Univ. Canada, (1 week);
- Hazel Everett, Univ. of Victoria, Canada (1 week), McGill Univ. Canada, (1 week);
- Sylvain Petitjean, Univ. of Sevilla, Spain (1 week);
- Guillaume Batog, i-MATH Winter School DocCourse Combinatorics and Geometry 2009: Discrete and Computational Geometry, Universitat Autònoma de Barcelona (2 weeks).

7. Dissemination

7.1. Teaching

All of the teaching activities were carried out in Nancy. The research Masters program is a joint degree with Univ. Nancy 1, Univ. Nancy 2 and the engineering school INPL. These three institutes are jointly known as University of Nancy.

Several members of the group, in particular the professors, assistant professors and Ph.D. students, actively teach at Université Nancy 2, Université Henri Poincaré Nancy 1, and INPL. Members of the group also teach in the Master of Computer Science of Nancy; namely H. Everett, contributed to the module "Modelisation of geometric data". X. Goaoc also intervenes in the Master's program of the geology school at INPL with lectures on the same topic.

7.2. Visibility

Program and Paper Committee:

- Hazel Everett: Program committee of the Canadian Conference on Computational Geometry, (CCCG'09). Program committee of the ACM Symposium on Computational Geometry 2010 (SoCG'10).
- Sylvain Petitjean: Program committee of the Shape Modeling International Conference (SMI'09). Paper committee of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR'09).

Editorial responsibilities:

- Hazel Everett: Editor of the *Journal of Computational Geometry*.
- Xavier Goaoc: Editor of the *Journal of Computational Geometry*.
- Sylvain Lazard: Guest editor (with L. Gonzalez-Vega, Santander Univ., Spain) of *Mathematics in Computer Science* (special issue on Computational Geometry and CAGD). Guest editor of *Computational Geometry: Theory and Applications* (special issue on selected papers from EuroCG'08).

Workshop Organizations:

- Hazel Everett and Sylvain Lazard co-organized with S. Whitesides (McGill & Victoria Universities) the 8th Workshop on Geometry Problems in Computer Graphics[1] (Bellairs Research Institute of McGill University) in Feb. (1 week).

Thesis and habilitation committee:

- Sylvain Petitjean: member of the habilitation committee of P. Alliez, Inria Sophia-Antipolis.

Participation to invitation-only workshops:

- Xavier Goaoc, Talk at the *Canada-Japan Workshop on Discrete and Computational Geometry,* Tokyo, Japan. Talk at the workshop *Transversal and Helly-Type Theorems in Geometry, Combinatorics and Topology,* BIRS, Banff, Canada.
- Hazel Everett and Sylvain Lazard, Talks at the *Dagstuhl Workshop on Computational Geometry,* Germany.

Other responsibilities:

- Hazel Everett: Director of the Mathematics and Computer Science Department of Université Nancy 2 (since 2006). Member of the *Équipe de direction* of LORIA (since 2007). President of the hiring committee for computer science, Université Nancy 2 (since 2008). Member of the hiring committee of LORIA (comipers-enseignants). IAEM representative in the Research Council of Nancy-Université (since 2007). LORIA representative in the Council of the Charles Hermite Research Federation. Coordinator of the CCD Section 27 (since 2008). Correspondent of University Nancy 2 for the computer science Master UHP/Nancy 2 (since 2007).
- Sylvain Lazard: Head of the INRIA Nancy-Grand Est *PhD and Post-doc hiring committee*. Member of the *Bureau du Département Informatique de Formation Doctorale* of the *École Doctorale IAE+M*. Reviewer for the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Dutch National Science Foundation.

[1] Workshop on Geometry Problems in Computer Graphics

- Laurent Dupont: Director of the departement *Services et réseaux de communication* of IUT Charlemagne, University Nancy 2 (since 2008).
- Xavier Goaoc: Member of the hiring committee for computer science, University Paris 6. *Correspondent Europe* of INRIA Nancy Grand-Est. Reviewer for the Israeli Science Foundation.
- Sylvain Petitjean: Scientific delegate of INRIA Nancy Grand-Est and chairman of its Project Committee (since 2009). Member of the *Équipes de direction* of LORIA and INRIA Nancy Grand-Est. Member of the AERES evaluation committee of LIENS, ENS Paris.
- Marc Pouget: Member of the CGAL Editorial Board (since 2008).

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