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Activity Report 2012

Project-Team COMMANDS

Control, Optimization, Models, Methods and Applications for Nonlinear Dynamical Systems

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP), Unité de Mathématiques Appliquées (UMA - ENSTA)

RESEARCH CENTER
Saclay - Île-de-France

THEME
Optimization and control of dynamic systems

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Project-Team COMMANDS

Keywords: System Analysis And Control, Nonlinear Control, Optimal Control, Numerical Methods, Nonsmooth Analysis

Team Commands is shared between the CMAP (Ecole Polytechnique) and the UMA (Ensta ParisTech).

Creation of the Project-Team: 2009 January 01.

1. Members

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2. Overall Objectives

2.1. Scientific directions

Commands is a team devoted to dynamic optimization, both for deterministic and stochastic systems. This includes the following approaches: trajectory optimization, deterministic and stochastic optimal control, stochastic programming, dynamic programming and Hamilton-Jacobi-Bellman equation.

Our aim is to derive new and powerful algorithms for solving numerically these problems, with applications in several industrial fields. While the numerical aspects are the core of our approach it happens that the study of convergence of these algorithms and the verification of their well-posedness and accuracy raises interesting and difficult theoretical questions, such as, for trajectory optimization: qualification conditions and second-order optimality condition, well-posedness of the shooting algorithm, estimates for discretization errors; for the Hamilton-Jacobi-Bellman approach: accuracy estimates, strong uniqueness principles when state constraints are present, for stochastic programming problems: sensitivity analysis.

2.2. Industrial impact

For many years the team members have been deeply involved in various industrial applications, often in the framework of PhD theses or of postdocs. The Commands team itself has dealt since its foundation in 2007 with several types of applications:

- Space vehicle trajectories, in collaboration with CNES, the French space agency,
- Production, management, storage and trading of energy resources (in collaboration with EDF, GDF and TOTAL).
- Energy management for hybrid vehicles (in collaboration with Renault).

We give more details in the Application domain section.

3. Research Program

3.1. Historical aspects

The roots of deterministic optimal control are the "classical" theory of the calculus of variations, illustrated by the work of Newton, Bernoulli, Euler, and Lagrange (whose famous multipliers were introduced in [69]), with improvements due to the "Chicago school", Bliss [44] during the first part of the 20th century, and by the notion of relaxed problem and generalized solution (Young [77]).

Trajectory optimization really started with the spectacular achievement done by Pontryagin's group [75] during the fifties, by stating, for general optimal control problems, nonlocal optimality conditions generalizing those of Weierstrass. This motivated the application to many industrial problems (see the classical books by Bryson and Ho [50], Leitmann [71], Lee and Markus [70], Ioffe and Tihomirov [66]). Since then, various theoretical achievements have been obtained by extending the results to nonsmooth problems, see Aubin [40], Clarke [51], Ekeland [58].

Dynamic programming was introduced and systematically studied by R. Bellman during the fifties. The HJB equation, whose solution is the value function of the (parameterized) optimal control problem, is a variant of the classical Hamilton-Jacobi equation of mechanics for the case of dynamics parameterized by a control variable. It may be viewed as a differential form of the dynamic programming principle. This nonlinear first-order PDE appears to be well-posed in the framework of *viscosity solutions* introduced by Crandall and Lions [53], [54], [52]. These tools also allow to perform the numerical analysis of discretization schemes. The theoretical contributions in this direction did not cease growing, see the books by Barles [42] and Bardi and Capuzzo-Dolcetta [41].

3.2. Trajectory optimization

The so-called *direct methods* consist in an optimization of the trajectory, after having discretized time, by a nonlinear programming solver that possibly takes into account the dynamic structure. So the two main problems are the choice of the discretization and the nonlinear programming algorithm. A third problem is the possibility of refinement of the discretization once after solving on a coarser grid.

In the *full discretization approach*, general Runge-Kutta schemes with different values of control for each inner step are used. This allows to obtain and control high orders of precision, see Hager [62], Bonnans [47]. In an interior-point algorithm context, controls can be eliminated and the resulting system of equation is easily solved due to its band structure. Discretization errors due to constraints are discussed in Dontchev et al. [57]. See also Malanowski et al. [72].

In the *indirect* approach, the control is eliminated thanks to Pontryagin's maximum principle. One has then to solve the two-points boundary value problem (with differential variables state and costate) by a single or multiple shooting method. The questions are here the choice of a discretization scheme for the integration of the boundary value problem, of a (possibly globalized) Newton type algorithm for solving the resulting finite dimensional problem in IR^n (n is the number of state variables), and a methodology for finding an initial point.

For state constrained problems or singular arcs, the formulation of the shooting function may be quite elaborate [45], [46], [39]. As initiated in [61], we focus more specifically on the handling of discontinuities, with ongoing work on the geometric integration aspects (Hamiltonian conservation).

3.3. Hamilton-Jacobi-Bellman approach

This approach consists in calculating the value function associated with the optimal control problem, and then synthesizing the feedback control and the optimal trajectory using Pontryagin's principle. The method has the great particular advantage of reaching directly the global optimum, which can be very interesting, when the problem is not convex.

Characterization of the value function From the dynamic programming principle, we derive a characterization of the value function as being a solution (in viscosity sense) of an Hamilton-Jacobi-Bellman equation, which is a nonlinear PDE of dimension equal to the number n of state variables. Since the pioneer works of Crandall and Lions [53], [54], [52], many theoretical contributions were carried out, allowing an understanding of the properties of the value function as well as of the set of admissible trajectories. However, there remains an important effort to provide for the development of effective and adapted numerical tools, mainly because of numerical complexity (complexity is exponential with respect to n).

Numerical approximation for continuous value function Several numerical schemes have been already studied to treat the case when the solution of the HJB equation (the value function) is continuous. Let us quote for example the Semi-Lagrangian methods [60], [59] studied by the team of M. Falcone (La Sapienza, Rome), the high order schemes WENO, ENO, Discrete galerkin introduced by S. Osher, C.-W. Shu, E. Harten [63], [64], [65], [73], and also the schemes on nonregular grids by R. Abgrall [38], [37]. All these schemes rely on finite differences or/and interpolation techniques which lead to numerical diffusions. Hence, the numerical solution is unsatisfying for long time approximations even in the continuous case.

One of the (nonmonotone) schemes for solving the HJB equation is based on the Ultrabee algorithm proposed, in the case of advection equation with constant velocity, by Roe [76] and recently revisited by Després-Lagoutière [56], [55]. The numerical results on several academic problems show the relevance of the antidiffusive schemes. However, the theoretical study of the convergence is a difficult question and is only partially done.

Optimal stochastic control problems occur when the dynamical system is uncertain. A decision typically has to be taken at each time, while realizations of future events are unknown (but some information is given on their distribution of probabilities). In particular, problems of economic nature deal with large uncertainties (on prices, production and demand). Specific examples are the portfolio selection problems in a market with risky and non-risky assets, super-replication with uncertain volatility, management of power resources (dams, gas). Air traffic control is another example of such problems.

Nonsmoothness of the value function. Sometimes the value function is smooth (e.g. in the case of Merton's portfolio problem, Oksendal [78]) and the associated HJB equation can be solved explicitly. Still, the value function is not smooth enough to satisfy the HJB equation in the classical sense. As for the deterministic case, the notion of viscosity solution provides a convenient framework for dealing with the lack of smoothness, see

Pham [74], that happens also to be well adapted to the study of discretization errors for numerical discretization schemes [67], [43].

Numerical approximation for optimal stochastic control problems. The numerical discretization of second order HJB equations was the subject of several contributions. The book of Kushner-Dupuis [68] gives a complete synthesis on the Markov chain schemes (i.e Finite Differences, semi-Lagrangian, Finite Elements, ...). Here a main difficulty of these equations comes from the fact that the second order operator (i.e. the diffusion term) is not uniformly elliptic and can be degenerated. Moreover, the diffusion term (covariance matrix) may change direction at any space point and at any time (this matrix is associated the dynamics volatility).

For solving stochastic control problems, we studied the so-called Generalized Finite Differences (GFD), that allow to choose at any node, the stencil approximating the diffusion matrix up to a certain threshold [49]. Determining the stencil and the associated coefficients boils down to a quadratic program to be solved at each point of the grid, and for each control. This is definitely expensive, with the exception of special structures where the coefficients can be computed at low cost. For two dimensional systems, we designed a (very) fast algorithm for computing the coefficients of the GFD scheme, based on the Stern-Brocot tree [48].

4. Application Domains

4.1. Introduction

Commands is a team with a strong commitment in tackling real-life applications in addition to theoretical challenges. This shows in our long history of contracts with industrial partners. In the recent years, we have mainly contributed to the following fields of application.

4.2. Aerospace applications

In the framework of a long-term partnership with the Cnes, and more recently Astrium, we have studied trajectory optimization for space launcher problems. This kind of problems typically involves hard constraints (thermal flux, mechanical efforts) and inexact models (atmosphere, aerodynamic forces). The two main achievments were to study when singular arcs may occur, and to show the effectiveness of a HJB approach on a reduced model. Singular arcs are flight phases with a non-maximal thrust, induced by a tradeoff between speed and atmospheric drag; they cause difficulties of both theoretical and practical nature. The latter point is the first step in the process of applying global methods to this class of difficult problems.

4.3. Trading applications

In a partnership with Total, we have studied problems dealing with the trading of Liquefied Natural Gas. We have computed maximizing revenue policies, by combining the Stochastic Dual Dynamic Programmin approach (SDDP) with a quantization method for the noise that enters in prices. We have also given partial results for the case of integer decision.

4.4. Energy applications

With Renault, we have studied problems of energy management for hybrid vehicles. Hybrid vehicles include an auxiliary thermal (gas) engine that is used as a range extender for the main electric propulsion. We are interested in determining the optimal policies for energy management, taking into account some stochatic incertainties, as well as execution delay and decision lags.

5. Software and Platforms

5.1. Bocop

Participants: Pierre Martinon [corresponding author], Vincent Grélard, Daphné Giorgi, Frédéric Bonnans.

Web page: http://bocop.org

The Bocop project aims to develop an open-source toolbox for solving optimal control problems, with collaborations with industrial and academic partners. Optimal control (optimization of dynamical systems governed by differential equations) has numerous applications in transportation, energy, process optimization, and biology. The software reuses some packages from the COIN-OR library, in particular the well-known nonlinear programming solver Ipopt, and also features a user-friendly interface.

The project is supported by Inria with the arrival of Vincent Grelard as developer in October 2010, and then Daphné Giorgi in October 2012. The first prototype was released in 2011, and version 1.4 is scheduled for the end of 2012. Bocop was first succesfully tested on several academic problems, see [28] available on http://bocop.org. In 2012, several research collaborations were initiated in fields such as bio-reactors for energy production (ref ECC), swimming micro-robots, and quantum control for medical imaging. Bocop was also featured during our participation in the Imatch "Optimisation and Control" in october.

5.2. CollAv

Participants: Hasnaa Zidani [corresponding author], Olivier Bokanowski, Anna Desilles.

This software simulates the evolution of controlled dynamical systems (possibly under uncertainties). The numerical algorithm here is based on HJB or viability approaches, and allows the design of optimal planning strategies (according to a criterion determined by the user: time, energy, ...). It also provides conflict resolution and avoidance of collisions with fixed or moving obstacles. So far, the software is used in collaboration with DGA for avodance collision of UaVs, and by Volkswagen in some studies related to collision avoidance of cars.

5.3. OCOPHyS

Participants: Hasnaa Zidani [corresponding author], Giovanni Granato.

This is a software for optimisation-based controller design for operating in different regimes or modes of operation. The software can be used, for example, to determine the optimal management for hybrid vehicles or hybrid engines with multiple energy sources. However, the methods used in software are still quite general and can be used in many applications.

5.4. BiNoPe-HJ

Participants: Hasnaa Zidani [corresponding author], Olivier Bokanowski, Anna Desilles, Jun-Yi Zhao.

Web page: http://www.ensta-paristech.fr/ zidani/BiNoPe-HJ

This project aims at developping sequential and parallel MPI/openMP C++ solvers for the approximation of Hamilton-Jacobi-Bellman (HJB) equations in a d-dimensional space. The main goal is to provide an HJB solvers that can work in dimension d (limited by the machine's capacity). The solver outputs can be visualized with Matlab or Paraview (via VTK files).

The development of the HJB Solver has been initiated under a partnership between COMMANDS and the SME HPC-project in the period between December 2009 to November 2011. Currently, it is still maintained and improved by COMMANDS.

In 2012, two versions were released:

- HJB-SEQUENTIAL-REF: sequential version that can run on any machine
- HJB-PARALLEL-REF: parallel version that can run only on multi-core architectures.

5.5. Shoot

Participant: Pierre Martinon [corresponding author].

Web page: http://www.cmap.polytechnique.fr/~martinon/codes.html

Shoot was designed for the resolution of optimal control problems via indirect methods (necessary conditions, Pontryagin's Maximum Principle). Such methods transform the original problem into finding a zero of a certain shooting function. The package offers several choices of integrators and solvers, and can handle control discontinuities. Features also include the use of variational equations to compute the Jacobian of the shooting function, as well as homotopy and grid shooting techniques for easier initialization.

6. New Results

6.1. Optimal control of partial differential equations

6.1.1. Optimal control of a semilinear parabolic equation with singular arcs

Participant: Frédéric Bonnans.

This paper, published as Inria report 8099 [25], develops a theory of singular arc, and the corresponding second order necessary and sufficient conditions, for the optimal control of a semilinear parabolic equation with scalar control applied on the r.h.s. We obtain in particular an extension of Kelley's condition, and the characterization of a quadratic growth property for a weak norm.

6.2. Trajectory optimization

6.2.1. First and second order optimality conditions for optimal control problems of state constrained integral equations

Participants: Frédéric Bonnans, Xavier Dupuis.

In this work performed with Constanza De La Vega (U. Buenos Aires), and published as Inria report 7961 [26], we deal with optimal control problems of integral equations, with initial-final and running state constraints. The order of a running state constraint is defined in the setting of integral dynamics, and we work here with constraints of arbitrary high orders. First and second-order necessary conditions of optimality are obtained, as well as second-order sufficient conditions.

6.2.2. Sensitivity analysis for relaxed optimal control problems with final-state constraints

Participants: Frédéric Bonnans, Laurent Pfeiffer.

In this work, performed with Oana Serea (U. Perpignan), and published as Inria report 7977 [27], we compute a second-order expansion of the value function of a family of relaxed optimal control problems with final-state constraints, parameterized by a perturbation variable. The sensitivity analysis is performed for controls that we call R-strong solutions. They are optimal solutions with respect to the set of feasible controls with a uniform norm smaller than a given R and having an associated trajectory in a small neighborhood for the uniform norm. In this framework, relaxation enables us to consider a wide class of perturbations and therefore to derive sharp estimates of the value function.

6.2.3. Sensitivity analysis for the outages of nuclear power plants

Participants: Frédéric Bonnans, Laurent Pfeiffer.

In this work, performed with Kengy Barty (EDF), and published as Inria report 7884 [24]. Nuclear power plants must be regularly shut down in order to perform refueling and maintenance operations. The scheduling of the outages is the first problem to be solved in electricity production management. It is a hard combinatorial problem for which an exact solving is impossible.

Our approach consists in modelling the problem by a two-level problem. First, we fix a feasible schedule of the dates of the outages. Then, we solve a low-level problem of optimization of elecricity production, by respecting the initial planning. In our model, the low-level problem is a deterministic convex optimal control problem.

Given the set of solutions and Lagrange multipliers of the low-level problem, we can perform a sensitivity analysis with respect to dates of the outages. The approximation of the value function which is obtained could be used for the optimization of the schedule with a local search algorithm.

6.2.4. Optimization of the anaerobic digestion of microalgae in a coupled process

Participant: Pierre Martinon.

In this work in collaboration with Terence Bayen (U. Monptellier) and Francis Mairet (Inria Sophia), submitted to ECC13 [30], we study the maximization of the production of methane in a bioreactor coupling an anaerobic digester and a culture of micro-algae limited by light. The decision parameter is the dilution rate which is chosen as a control, and we enforce periodic constraints in order to repeat the same operation every day. The system is gathered into a three-dimensional system taking into account a day-night model of the light in the culture of micro-algae. Applying Pontryagin maximum principle, the necessary conditions on optimal trajectories indicate that the control consists of bang and/or singular arcs. We provide numerical simulations by both direct and indirect methods, which show the link between the light model and the structure of optimal solutions.

6.3. Stochastic programming

6.3.1. Solving multi-stage stochastic mixed integer linear programs by the dual dynamic programming approach

Participants: Frédéric Bonnans, Zhihao Cen.

In this work performed in the framework of the PhD thesis of Zhihao Cen, and published as an Inria report RR-7868 [29], We consider a model of medium-term commodity contracts management. Randomness takes place only in the prices on which the commodities are exchanged, whilst state variable is multi-dimensional, and decision variable is integer. In our previous article, we proposed an algorithm based on the quantization of random process and a dual dynamic programming type approach to solve the continuous relaxation problem. In this paper, we study the multi-stage stochastic mixed integer linear program (SMILP) and show the difficulty when using dual programming type algorithm. We propose an approach based on the cutting plane method combined with the algorithm in our previous article, which gives an upper and a lower bound of the optimal value and a sub-optimal integer solution. Finally, a numerical test on a real problem in energy market is performed.

6.3.2. Two methods of pruning Benders' cuts and their application to the management of a gas portfolio

Participant: Laurent Pfeiffer.

This report, coauthored with R. Apparigliato and S. Auchapt (Gdf Suez), and published as Inria report 8133 [31], describes a gas portfolio management problem, which is solved with the SDDP (Stochastic Dual Dynamic Programming) algorithm. We present some improvements of this algorithm and focus on methods of pruning Benders' cuts, that is to say, methods of picking out the most relevant cuts among those which have been computed. Our territory algorithm allows a quick selection and a great reduction of the number of cuts. Our second method only deletes cuts which do not contribute to the approximation of the value function, thanks to a test of usefulness. Numerical results are presented.

6.4. Hamilton-Jacobi approach

6.4.1. Hamilton-Jacobi equations in singular domains

Participants: Zhiping Rao, Hasnaa Zidani.

A good deal of attention has been devoted to the analysis of Hamilton–Jacobi equations adapted to unconventional domains, particularly in view of application to control problems and traffic models. The topic is new and capable of interesting developments, the results so far obtained have allowed to clarify under reasonable assumptions, basic items as the right notion of viscosity solution to be adopted and the validity of comparison principles.

- The work [19], co-authored with C. Imbert (LAMA, U. Paris-Est) and R. Monneau (Cermics, ENPC), focuses on a Hamilton-Jacobi approach to junction problems with applications to traffic flows. More specifically, the paper is concerned with the study of a model case of

  first order Hamilton-Jacobi equations posed on a *junction*, that is to say the union of a finite number of half-lines with a unique common point. The main result is a comparison principle. We also prove existence and stability of solutions. The two challenging difficulties are the singular geometry of the domain and the discontinuity of the Hamiltonian. As far as discontinuous Hamiltonians are concerned, these results seem to be new. They are applied to the study of some models arising in traffic flows. The techniques developed here provide new powerful tools for the analysis of such problems.
- This work deals with deterministic control problems where the dynamic can be completely different in multi-complementary domains of the space IR^d. As a consequence, the dynamics present discontinuities at the interfaces of these domains. This leads to a complex interplay that has to be analyzed among transmission conditions to "glue" the propagation of the value function on the interfaces. Several questions arise: how to define properly the value function and what is the right Bellman Equation associated to this problem?. In the case of finite horizon problems without runing cost, a jonction condition is derived on the interfaces, and a precise viscosity notion is provided in a paper in progress. Moreover, a uniqueness result of a viscosity solution is shown.

6.4.2. A general Hamilton-Jacobi framework for nonlinear state-constrained control problems

Participants: Olivier Bokanowski, Hasnaa Zidani.

This work [10], co-authored with Albert Altarovici, deals with deterministic optimal control problem with state constraints and nonlinear dynamics. It is known for such a problem that the value function is in general discontinuous and its characterization by means of an HJ equation requires some controllability assumptions involving the dynamics and the set of state constraints. Here, we first adopt the viability point of view and look at the value function as its epigraph. Then, we prove that this epigraph can always be described by an auxiliary optimal control problem free of state constraints, and for which the value function is Lipschitz continuous and can be characterized, without any additional assumptions, as the unique viscosity solution of a Hamilton-Jacobi equation. The idea introduced in this paper bypasses the regularity issues on the value function of the constrained control problem and leads to a constructive way to compute its epigraph by a large panel of numerical schemes. Our approach can be extended to more general control problems. We study in this paper the extension to the infinite horizon problem as well as for the two-player game setting. Finally, an illustrative numerical example is given to show the relevance of the approach.

6.4.3. State-constrained optimal control problems of impulsive differential equations

Participants: Nicolas Forcadel, Zhiping Rao, Hasnaa Zidani.

The research report [35] presents a study on optimal control problems governed by measure driven differential systems and in presence of state constraints. The first result shows that using the graph completion of the measure, the optimal solutions can be obtained by solving a reparametrized control problem of absolutely continuous trajectories but with time-dependent state-constraints. The second result shows that it is possible to characterize the epigraph of the reparametrized value function by a Hamilton-Jacobi equation without assuming any controllability assumption

6.4.4. Level-set approach for reachability analysis of hybrid systems under lag constraints

Participants: Giovanni Granato, Hasnaa Zidani.

The study in [36] aims at characterizing a reachable set of a hybrid dynamical system with a lag constraint in the switch control. The setting does not consider any controllability assumptions and uses a level-set approach. The approach consists in the introduction of an adequate hybrid optimal control problem with lag constraints on the switch control whose value function allows a characterization of the reachable set. The value function is in turn characterized by a system of quasi-variational inequalities (SQVI). We prove a comparison principle for the SQVI which shows uniqueness of its solution. A class of numerical finite differences schemes for solving the system of inequalities is proposed and the convergence of the numerical solution towards the value function is studied using the comparison principle. Some numerical examples illustrating the method are presented. Our study is motivated by an industrial application, namely, that of range extender electric vehicles. This class of electric vehicles uses an additional module *the range extender* as an extra source of energy in addition to its main source a high voltage battery. The methodolgy presented in [36] is used to establish the maximum range of a Hybrid vehicle, see [22].

6.5. Collision avoidance and motion planning

6.5.1. Collision analysis for a UAV

Participants: Anna Désilles, Hasnaa Zidani.

The *Sense and Avoid* capacity of Unmanned Aerial Vehicles (UAV) is one of the key elements to open the access to airspace for UAVs. In order to replace a pilot's *See and Avoid* capacity such a system has to be certified "as safe as a human pilot on-board". The problem is to prove that an unmanned aircraft equipped with a S&A system can comply with the actual air transportation regulations. A paper in progress aims to provide mathematical and numerical tools to link together the safety objectives and sensors specifications. Our approach starts with the natural idea of a specified "safety volume" around the aircraft: the safety objective is to guarantee that no other aircraft can penetrate this volume. We use a general reachability and viability concepts to define nested sets which are meaningful to allocate sensor performances and manoeuvring capabilities necessary to protect the safety volume. Using the general framework of HJB equations for the optimal control and differential games, we give a rigorous mathematical characterization of these sets. Our approach allows also to take into account some uncertainties in the measures of the parameters of the incoming traffic. We also provide numerical tools to compute the defined sets, so that the technical specifications of a S&A system can be derived in accordance with a small set of intuitive parameters. We consider several dynamical models corresponding to the different choices of maneuvers (lateral, longitudinal and mixed). Our numerical simulations show clearly that the nature of used maneuvers is an important factor in the specifications of sensor's performances.

6.6. Numerical methods for HJ equations

6.6.1. An adaptive sparse grid semi-lagrangian scheme for first order Hamilton-Jacobi Bellman equations

Participant: Olivier Bokanowski.

The paper [14], co-authored with M. Griebel (Fraunhofer SCAI & Univ. Bonn), J. Garcke and I. Klopmpaker (TUB, Berlin) proposes a semi-Lagrangian scheme using a spatially adaptive sparse grid to deal with non-linear time-dependent Hamilton-Jacobi Bellman equations. We focus in particular on front propagation models in higher dimensions which are related to control problems. We test the numerical efficiency of the method on several benchmark problems up to space dimension $d = 8$, and give evidence of convergence towards the exact viscosity solution. In addition, we study how the complexity and precision scale with the dimension of the problem.

6.6.2. A discontinuous Galerkin scheme for front propagation with obstacles

Participant: Olivier Bokanowski.

In [33], co-authored with C.-W. Shu (Brown Univ.) and Y. Cheng (Michigan Univ.), some front propagation problems in the presence of obstacles are analysed. We extend a previous work (Bokanowski, Cheng and Shu, SIAM J. Scient. Comput., 2011), to propose a simple and direct discontinuous Galerkin (DG) method adapted to such front propagation problems. We follow the formulation of (Bokanowski, Forcadel and Zidani, SIAM J. Control Optim. 2010), leading to a level set formulation driven by $\min(u_t + H(x, \nabla u), u - g(x)) = 0$, where $g(x)$ is an obstacle function. The DG scheme is motivated by the variational formulation when the Hamiltonian H is a linear function of ∇u, corresponding to linear convection problems in the presence of obstacles. The scheme is then generalized to nonlinear equations, written in an explicit form. Stability analysis is performed for the linear case with Euler forward, a Heun scheme and a Runge-Kutta third order time discretization using the technique proposed in (Zhang and Shu, SIAM J. Control and Optim., 2010). Several numerical examples are provided to demonstrate the robustness of the method. Finally, a narrow band approach is considered in order to reduce the computational cost.

6.6.3. Semi-Lagrangian discontinuous Galerkin schemes for some first and second order PDEs

Participant: Olivier Bokanowski.

Explicit, unconditionally stable, high order schemes for the approximation of some first and second order linear, time-dependent partial differential equations (PDEs) are proposed in [34], in collaboration with G. Simarmata (internship 2011, currently in RI dep. of Rabobank). The schemes are based on a weak formulation of a semi-Lagrangian scheme using discontinuous Galerkin elements. It follows the ideas of the recent works of Crouseilles, Mehrenberger and Vecil (2010) and of Qiu and Shu (2011), for first order equations, based on exact integration, quadrature rules, and splitting techniques. In particular we obtain high order schemes, unconditionally stable and convergent, in the case of linear second order PDEs with constant coefficients. In the case of non-constant coefficients, we construct "almost" unconditionally stable second order schemes and give precise convergence results. The schemes are tested on several academic examples, including the Black and Scholes PDE in finance.

7. Bilateral Contracts and Grants with Industry

7.1. Renault

Participants: Frédéric Bonnans, Giovanni Granato, Hasnaa Zidani.

This contract has supported the PhD thesis of Giovanni Granato. The purpose of this collaboration is to apply optimal control techniques to enhance the performance of the power management of hybrid vehicles. More precisely, the techniques concerned are viscosity solutions of Hamilton-Jacobi (HJ) equations, level set methods in reachability analysis, stochastic dynamic programming (SDP), stochastic dual dynamic programming (SDDP) and chance constrained optimal control. These are relatively sophisticated optimal control techniques, presenting a rupture (from the application point of view) of more classical techniques (e.g. dynamic programming, maximum principle and heuristic algorithms) found in the literature. The outcome of the PhD work is to assess the general interest in applying such techniques to the power management of hybrid vehicles. This includes stating the relevant modeling choices, implementing a research-level code of the algorithms for simulations and providing a proper interpretation of the simulations results.

The research undertaken in this contract have lead to four submitted patents (Renault-Inria), a numerical platform for simulations of the studied technics, 1 accepted conference paper (CDC), 2 submitted papers in peer-reviewed journals, and 2 preprints in preparation.

7.2. Astrium-Eads

Participant: Hasnaa Zidani.

This collaboration aims at analysing the sensitivity properties of a trajectory optimisation problem under probabilistic constraints (on modelling errors, component failure, ...etc). This includes a modeling of the problem, and implementation of efficient algorithms supportting the theoretical study. The collaboration started in 2012 and will last three years.

8. Partnerships and Cooperations

8.1. National Initiatives

8.1.1. DGA

Participants: Olivier Bokanowski, Anna Désilles, Hasnaa Zidani.

Our team has a financial support from the DGA, within the programme "etudes Laboratoires". The research programme concerns the Hamilton-Jacobi approach for optimal control problems with state constraints. Our main interest in this class of control problems comes from the fact that the field has an important potential role in future technological developments to take account of environmental, physical or economical constraints.

A part of our findings in this topic have been used to develop a software for collision avoidance of a Uav.

8.2. European Initiatives

8.2.1. FP7 Projects

8.2.1.1. SADCO

Title: Sensitivity Analysis for Deterministic Controller Design

Instrument: Initial Training Network (ITN)

Duration: January 2011 - December 2014

Coordinator: Inria (France)

Others partners: Univ. of Louvain, Univ. Bayreuth, Univ. Porto, Univ. Rome - La Sapienza, ICL, Astrium-Eads, Astos solutions, Volkswagen, Univ. Padova, Univ. Pierre et Marie Curie

See also: http://itn-sadco.inria.fr

Abstract: Optimisation-based control systems concern the determination of control strategies for complex, dynamic systems, to optimise some measures of best performance. It has the potential for application to a wide range of fields, including aerospace, chemical processing, power systems control, transportation systems and resource economics. It is of special relevance today, because optimization provides a natural framework for determining control strategies, which are energy efficient and respect environmental constraints. The multi-partner initial training network SADCO aims at: Training young researchers and future scientific leaders in the field of control theory with emphasis on two major themes sensitivity of optimal strategies to changes in the optimal control problem specification, and deterministic controller design; Advancing the theory and developing new numerical methods; Conveying fundamental scientific contributions within European industrial sectors.

8.2.2. Collaborations with Major European Organizations

Univ. Rome 1 - La Sapienza: Department of Mathematics

Collaboration with Antonio Siconolfi on "Hamilton-Jacobi equations in multi-domains".

Univ. Rome 2: Department of Mathematics

Numerical schemes for Hamilton-Jacobi coupled systems, controller design for hybrid systems.

8.3. International Initiatives

8.3.1. Inria Associate Teams

8.3.1.1. OCONET

Title: Optimization and control in network economics

Inria principal investigator: J.F. Bonnans

International Partner (Institution - Laboratory - Researcher):

University of Chile (Chile) - Center for Mathematical Modeling - Alejandro Jofre

Duration: 2012 - 2014

Web page: http://www.cmm.uchile.cl/EA_OCONET

Limited resources in telecommunication, energy, gas and water supply networks, lead to multi-agent interactions that can be seen as games or economic equilibrium involving stochastic optimization and optimal control problems. Interaction occurs within a network, where decisions on what to produce, consume, trade or plan, are subject to constraints imposed by node and link capacities, risk, and uncertainty, e.g. the capacity of generators and transmission lines; capacity of pipeline in gas supply; switches and antennas in telecommunication. At the same time, nonlinear phenomena arise from price formation as a consequence of demand-supply equilibria or multi-unit auction processes in the case of energy and telecommunication. We will focus first in this project in electricity markets in which there are producers/consumers PCs, and an agent called ISO (Independent system operator) in charge of the management of the network. One major application we have in mind is the one of smart (electrical) grids, in view of the increased use of renewable energies, that is, a massive entry of wind, geothermal, solar in particular.

8.3.2. Inria International Partners

Univ. Buenos Aires: Department of Mathematics

Collaboration with Constanza de la Vega on the optimal control of systems with delay.

Moscow State Univ.: Department of Mathematics

Collaboration with Andrei Dmitruk on optimal control with singular arcs.

ENIT, Tunis: Department of Mathematics

Collaboration with Mohamed Mnif on the numerical methods for swing options.

Louisiane State University, USA

Collaboration with Peter Wolenski on stratified controlled systems.

8.3.3. Participation In International Programs

The team is involved in the "Energy Optimization" group of the Inria research center in Chile (CIRIC). Several visits to Chile were conducted in relation with this project.

8.4. International Research Visitors

8.4.1. Visits of International Scientists

- Claudia Sagastizabal from IMPA in Rio (2 weeks, November 2012)
- Fabio Ancona, from Univ. of Padova (1 week, October 2012)
- Roberto Ferretti, from Univ. of Rome II (2 weeks, October 2012)
- Antonio Siconolfi, from Univ. of Rome I (2 weeks, June 2012)
- Lars Grüne, from Univ. of Bayreuth (1 week, June 2012)
- Adam Oberman, from Univ. of Vancouver (2 weeks, May 2012)
- Peter Wolenski, from Univ. of Louisiane (3 days, March 2012)

- Mohamed Mnif, from ENIT (2 weeks, February 2012)

8.4.1.1. Internships

Imene BEN LATIFA (from Feb 2012 until May 2012)

Subject: Numerical computation of swing options

Institution: Ecole Nationale d'Ingénieurs de Tunis (Tunisia)

Lucas Corrales (from May 2012 until Jul 2012)

Subject: Optimal control for some drug models

Institution: National University of the Center of the Buenos Aires Province (Argentina)

8.4.2. Visits to International Teams

- Olivier Bokanowski visited the Mathematics Department at Brown Univ., for 1 week.
- Hasnaa Zidani visited the Mathematics Department at Univ. of Rome 1- La sapienza, for 1 week.
- Olivier Bokanowski visited the Mathematical institute (Oxford), for 1 week.

9. Dissemination

9.1. Scientific Animation

9.1.1. Editorial boards, scientific societies

- F. Bonnans is Corresponding Editor of "ESAIM:COCV" (Control, Optimization and Calculus of Variations), and Associate Editor of "Applied Mathematics and Optimization", "Optimization, Methods and Software", and "Series on Mathematics and its Applications, Annals of The Academy of Romanian Scientists".
- F. Bonnans is chairman of the SMAI-MODE group (the optimization group of the French Applied Mathematics Society).

9.1.2. Organization of conferences

The team members have been involved in the organisation of several meetings and conferences, within the SADCO project (http://itn-sadco.inria.fr). In particular, F. Bonnans and H. Zidani organised the workshop "Applied and Numerical Optimal Control", in Paris.

9.2. Teaching - Supervision - Juries

9.2.1. Teaching

F. Bonnans: Optimal control, 7h, M2, Ensta, France.

F. Bonnans: Continuous Optimization, 18h, M2, Ecole Polytechnique and U. Paris 6, France.

F. Bonnans: Numerical analysis of partial differential equations arising in finance and stochastic control, 24h, M2, Ecole Polytechnique and U. Paris 6, France.

H. Zidani: Optimal control, 14h, M2, Ensta, France.

H. Zidani: Numerical methods for front propagation, 21h, M2, Ensta France

9.2.2. Supervision

PhD & HdR

PhD: Giovanni Granato, Energy management for an electric vehicle with range extender. December 10, 2012, H. Zidani and F. Bonnans.

PhD in progress : Imène Ben-Latifa, Optimal multiple stopping and valuation of swing options in jump models. Oct. 2010, F. Bonnans and M. Mnif (ENIT, Tunis).

PhD in progress : Xavier Dupuis, Optimal control of populations; medical applications. Sept. 2010, F. Bonnans.

PhD in progress : Laurent Pfeiffer, Optimal control of large electrical networks. Sept. 2010, F. Bonnans.

PhD in progress: Zhiping Rao, Hamilton-Jacobi equations with discontinuous coefficients. Sept. 2010, H. Zidani and N. Forcadel.

PhD in progress: Athena Picarelli, First and Second Order Hamilton-Jacobi equations for State-Constrained Control Problems. Nov. 2011, O. Bokanowski and H. Zidani

PhD in progress: Cristopher Hermosilla, Feedback controls and optimal trajectories. Nov. 2011, H. Zidani.

PhD in progress: Mohamed Assellaou, Reachability analysis for stochastic controlled systems. Oct. 2011, O. Bokanowski and H.Zidani.

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Major publications by the team in recent years

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Publications of the year

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