



Activity Report 2014

Team TONUS

TOkamaks and NUmerical Simulations

RESEARCH CENTER
Nancy - Grand Est

THEME
**Earth, Environmental and Energy
Sciences**

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Team TONUS

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Creation of the Team: 2012 January 01.

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2. Overall Objectives

2.1. Overall Objectives

TONUS started in January 2014. It is a team of the Inria Nancy-Grand Est center. It is located in the mathematics institute (IRMA) of the university of Strasbourg.

The International Thermonuclear Experimental Reactor (ITER) is a large-scale scientific experiment that aims to demonstrate that it is possible to produce energy from fusion, by confining a very hot hydrogen plasma inside a toroidal chamber, called tokamak. In addition to physics and technology research, tokamak design also requires mathematical modeling and numerical simulations on supercomputers.

The objective of the TONUS project is to deal with such mathematical and computing issues. We are mainly interested in kinetic and gyrokinetic simulations of collisionless plasmas. In the TONUS project-team we are working on the development of new numerical methods devoted to such simulations. We investigate several classical plasma models, study new reduced models and new numerical schemes adapted to these models. We implement our methods in two software projects: Selalib¹ and CLAC² adapted to new computer architectures. We intend to run challenging simulations on high performance computers with thousands of nodes.

We have strong relations with the CEA-IRFM team and participate to the development of their gyrokinetic simulation software GYSELA. We are involved into two Inria Project Labs, respectively devoted to tokamak mathematical modeling and high performance computing on future exascale super-computers.

Finally, our subjects of interest are at the interaction between mathematics, computer science, High Performance Computing, physics and practical applications.

3. Research Program

3.1. Kinetic models for plasmas

The fundamental model for plasma physics is the coupled Vlasov-Maxwell kinetic model: the Vlasov equation describes the distribution function of particles (ions and electrons), while the Maxwell equations describe the electromagnetic field. In some applications, it may be necessary to take into account relativistic particles, which lead to consider the relativistic Vlasov equation, but generally, tokamak plasmas are supposed to be non relativistic. The particles distribution function depends on seven variables (three for space, three for velocity and one for time), which yields a huge amount of computations.

To these equations we must add several types of source terms and boundary conditions for representing the walls of the tokamak, the applied electromagnetic field that confines the plasma, fuel injection, collision effects, etc.

Tokamak plasmas possess particular features, which require developing specialized theoretical and numerical tools.

Because the magnetic field is strong, the particle trajectories have a very fast rotation around the magnetic field lines. A full resolution would require prohibitive amount of calculations. It is then necessary to develop models where the cyclotron frequency tends to infinity in order to obtain tractable calculations. The resulting model is called a gyrokinetic model. It allows us to reduce the dimensionality of the problem. Such models are implemented in GYSELA and Selalib. Those models require averaging of the acting fields during a rotation period along the trajectories of the particles. This averaging is called the gyroaverage and requires specific discretizations.

The tokamak and its magnetic fields present a very particular geometry. Some authors have proposed to return to the intrinsic geometrical versions of the Vlasov-Maxwell system in order to build better gyrokinetic models and adapted numerical schemes. This implies the use of sophisticated tools of differential geometry: differential forms, symplectic manifolds, and hamiltonian geometry.

In addition to theoretical modeling tools, it is necessary to develop numerical schemes adapted to kinetic and gyrokinetic models. Three kinds of methods are studied in TONUS: Particle-In-Cell (PIC) methods, semi-Lagrangian and fully Eulerian approaches.

¹<http://selalib.gforge.inria.fr/>

²<http://clac.gforge.inria.fr>

3.1.1. Gyrokinetic models: theory and approximation

In most phenomena where oscillations are present, we can establish a three-model hierarchy: (i) the model parameterized by the oscillation period, (ii) the limit model and (iii) the Two-Scale model, possibly with its corrector. In a context where one wishes to simulate such a phenomenon where the oscillation period is small and where the oscillation amplitude is not small, it is important to have numerical methods based on an approximation of the Two-Scale model. If the oscillation period varies significantly over the domain of simulation, it is important to have numerical methods that approximate properly and effectively the model parameterized by the oscillation period and the Two-Scale model. Implemented Two-Scale Numerical Methods (for instance by Frénod et al. [36]) are based on the numerical approximation of the Two-Scale model. These are called of order 0. A Two-Scale Numerical Method is called of order 1 if it incorporates information from the corrector and from the equation to which this corrector is a solution. If the oscillation period varies between very small values and values of order 1, it is necessary to have new types of numerical schemes (Two-Scale Asymptotic Preserving Schemes of order 1 or TSAPS) with the property being able to preserve the asymptotics between the model parameterized by the oscillation period and the Two-Scale model with its corrector. A first work in this direction has been initiated by Crouseilles et al. [32].

3.1.2. Semi-Lagrangian schemes

The Strasbourg team has a long and recognized experience in numerical methods of Vlasov-type equations. We are specialized in both particle and phase space solvers for the Vlasov equation: Particle-in-Cell (PIC) methods and semi-Lagrangian methods. We also have a longstanding collaboration with the CEA of Cadarache for the development of the GYSELA software for gyrokinetic tokamak plasmas.

The Vlasov and the gyrokinetic models are partial differential equations that express the transport of the distribution function in the phase space. In the original Vlasov case, the phase space is the six-dimension position-velocity space. For the gyrokinetic model, the phase space is five-dimensional because we consider only the parallel velocity in the direction of the magnetic field and the gyrokinetic angular velocity instead of three velocity components.

A few years ago, Eric Sonnendrücker and his collaborators introduce a new family of methods for solving transport equations in the phase space. This family of methods are the semi-Lagrangian methods. The principle of these methods is to solve the equation on a grid of the phase space. The grid points are transported with the flow of the transport equation for a time step and interpolated back periodically onto the initial grid. The method is then a mix of particle Lagrangian methods and eulerian methods. The characteristics can be solved forward or backward in time leading to the Forward Semi-Lagrangian (FSL) or Backward Semi-Lagrangian (BSL) schemes. Conservative schemes based on this idea can be developed and are called Conservative Semi-Lagrangian (CSL).

GYSELA is a 5D full gyrokinetic code based on a classical backward semi-Lagrangian scheme (BSL) [43] for the simulation of core turbulence that has been developed at CEA Cadarache in collaboration with our team [37]. Although GYSELA was carefully developed to be conservative at lowest order, it is not exactly conservative, which might be an issue when the simulation is under-resolved, which always happens in turbulence simulations due to the formation of vortices which roll up.

3.1.3. PIC methods

Historically PIC methods have been very popular for solving the Vlasov equations. They allow solving the equations in the phase space at a relatively low cost. The main disadvantage of the method is that, due to its random aspect, it produces an important numerical noise that has to be controlled in some way, for instance by regularizations of the particles, or by divergence correction techniques in the Maxwell solver. We have a longstanding experience in PIC methods and we started implement them in SeLaLib. An important aspect is to adapt the method to new multicore computers. See the work by Crestetto and Helluy [31].

3.2. Reduced kinetic models for plasmas

As already said, kinetic plasmas computer simulations are very intensive, because of the gyrokinetic turbulence. In some situations, it is possible to make assumptions on the shape of the distribution function that simplify the model. We obtain in this way a family of fluid or reduced models.

Assuming that the distribution function has a Maxwellian shape, for instance, we obtain the MagnetoHydro-Dynamic (MHD) model. It is physically valid only in some parts of the tokamak (at the edges for instance). The fluid model is generally obtained from the hypothesis that the collisions between particles are strong. At Inria, fine collision models are mainly investigated in the KALIFFE team. In our approach we do not assume that the collisions are strong, but rather try to adapt the representation of the distribution function according to its shape, keeping the kinetic effects. The reduction is not necessarily a consequence of collisional effects. Indeed, even without collisions, the plasma may still relax to an equilibrium state over sufficiently long time scales (Landau damping effect). Recently, a team at the Plasma Physics Institut (IPP) in Garching has carried out a statistical analysis of the 5D distribution functions obtained from gyrokinetic tokamak simulations [38]. They discovered that the fluctuations are much higher in the space directions than in the velocity directions (see Figure 1).

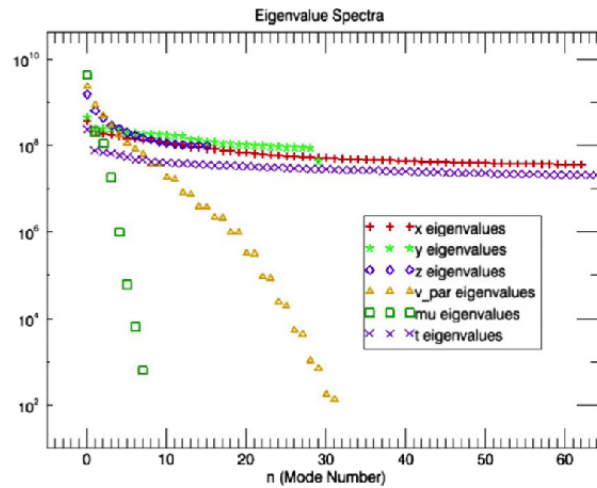


Figure 1. Space and velocity fluctuations spectra (from [38])

This indicates that the approximation of the distribution function could require fewer data while still achieving a good representation, even in the collisionless regime.

Our approach is different from the fluid approximation. In what follows we call this the “reduced model” approach. A reduced model is a model where the explicit dependence on the velocity variable is removed. In a more mathematical way, we consider that in some regions of the plasma, it is possible to exhibit a (preferably small) set of parameters α that allows us to describe the main properties of the plasma with a generalized “Maxwellian” M . Then

$$f(x, v, t) = M(\alpha(x, t), v).$$

In this case it is sufficient to solve for $\alpha(x, t)$. Generally, the vector α is solution of a first order hyperbolic system.

Several approaches are possible: waterbag approximations, velocity space transforms, *etc.*

3.2.1. Velocity space transformations

An experiment made in the 60's [41] exhibits in a spectacular way the reversible nature of the Vlasov equations. When two perturbations are applied to a plasma at different times, at first the plasma seems to damp and reach an equilibrium. But the information of the perturbations is still here and "hidden" in the high frequency microscopic oscillations of the distribution function. At a later time a resonance occurs and the plasma produces an echo. The time at which the echo occurs can be computed (see Villani ³, page 74). The fine mathematical study of this phenomenon allowed C. Villani and C. Mouhot to prove their famous result on the rigorous nonlinear Landau damping [42].

More practically, this experiment and its theoretical framework show that it is interesting to represent the distribution function by an expansion on an orthonormal basis of oscillating functions in the velocity variables. This representation allows a better control of the energy transfer between the low frequencies and the high frequencies in the velocity direction, and thus provides more relevant numerical methods. This kind of approach is studied for instance by Eliasson in [34] with the Fourier expansion.

In long time scales, filamentation phenomena result in high frequency oscillations in velocity space that numerical schemes cannot resolve. For stability purposes, most numerical schemes contain dissipation mechanisms that may affect the precision of the finest oscillations that could be resolved.

3.2.2. Adaptive modeling

Another trend in scientific computing is to optimize the computation time through adaptive modeling. This approach consists in applying the more efficient model locally, in the computational domain, according to an error indicator. In tokamak simulations, this kind of approach could be very efficient, if we are able to choose locally the best intermediate kinetic-fluid model as the computation runs. This field of research is very promising. It requires developing a clever hierarchy of models, rigorous error indicators, versatile software architecture, and algorithms adapted to new multicore computers.

3.2.3. Numerical schemes

As previously indicated, an efficient method for solving the reduced models is the Discontinuous Galerkin (DG) approach. It is possible to make it of arbitrary order. It requires limiters when it is applied to nonlinear PDEs occurring for instance in fluid mechanics. But the reduced models that we intent to write are essentially linear. The nonlinearity is concentrated in a few coupling source terms.

In addition, this method, when written in special set of variables, called the entropy variables, has nice properties concerning the entropy dissipation of the model. It opens the door to constructing numerical schemes with good conservation properties and no entropy dissipation, as already used for other systems of PDEs [44], [30], [40], [39].

3.3. Electromagnetic solvers

A precise resolution of the electromagnetic fields is essential for proper plasma simulation. Thus it is important to use efficient solvers for the Maxwell systems and its asymptotics: Poisson equation and magnetostatics.

The proper coupling of the electromagnetic solver with the Vlasov solver is also crucial for ensuring conservation properties and stability of the simulation.

Finally plasma physics implies very different time scales. It is thus very important to develop implicit Maxwell solvers and Asymptotic Preserving (AP) schemes in order to obtain good behavior on long time scales.

3.3.1. Coupling

The coupling of the Maxwell equations to the Vlasov solver requires some precautions. The most important is to control the charge conservation errors, which are related to the divergence conditions on the electric and magnetic fields. We will generally use divergence correction tools for hyperbolic systems presented for instance in [29] (and included references).

³Landau damping. CEMRACS 2010 lectures. <http://smi.emath.fr/cemracs/cemracs10/PROJ/Villani-lectures.pdf>

3.3.2. Implicit solvers

As already pointed out, in a tokamak, the plasma presents several different space and time scales. It is not possible in practice to solve the initial Vlasov-Maxwell model. It is first necessary to establish asymptotic models by letting some parameters (such as the Larmor frequency or the speed of light) tend to infinity. This is the case for the electromagnetic solver and this requires implementing implicit time solvers in order to efficiently capture the stationary state, the solution of the magnetic induction equation or the Poisson equation.

4. Application Domains

4.1. Controlled fusion and ITER

The search for alternative energy sources is a major issue for the future. Among others, controlled thermonuclear fusion in a hot hydrogen plasma is a promising possibility. The principle is to confine the plasma in a toroidal chamber, called a tokamak, and to attain the necessary temperatures to sustain nuclear fusion reactions. The International Thermonuclear Experimental Reactor (ITER) is a tokamak being constructed in Cadarache, France. This was the result of a joint decision by an international consortium made of the European Union, Canada, USA, Japan, Russia, South Korea, India and China. ITER is a huge project. As of today, the budget is estimated at 20 billion euros. The first plasma shot is planned for 2020 and the first deuterium-tritium operation for 2027.

Many technical and conceptual difficulties have to be overcome before the actual exploitation of fusion energy. Consequently, much research has been carried out around magnetically confined fusion. Among these studies, it is important to carry out computer simulations of the burning plasma. Thus, mathematicians and computer scientists are also needed in the design of ITER. The reliability and the precision of numerical simulations allow a better understanding of the physical phenomena and thus would lead to better designs. TONUS's main involvement is in such research.

The required temperatures to attain fusion are very high, of the order of a hundred million degrees. Thus it is imperative to prevent the plasma from touching the tokamak inner walls. This confinement is obtained thanks to intense magnetic fields. The magnetic field is created by poloidal coils, which generate the toroidal component of the field. The toroidal plasma current also induces a poloidal component of the magnetic field that twists the magnetic field lines (see Figure 2). The twisting is very important for the stability of the plasma. The idea goes back to research by Tamm and Sakharov, two Russian physicists, in the 50's.

Other devices are essential for the proper operation of the tokamak: divertor for collecting the escaping particles, microwave heating for reaching higher temperatures, fuel injector for sustaining the fusion reactions, toroidal coils for controlling instabilities, *etc.*

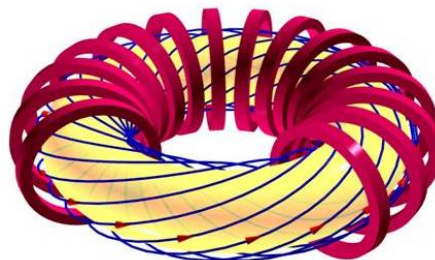


Figure 2. Poloidal coils and magnetic field lines geometry inside a tokamak

4.2. Other applications

The software and numerical methods that we develop can also be applied to other fields of physics or of engineering.

- For instance, we have a collaboration with the company AxesSim in Strasbourg for the development of efficient Discontinuous Galerkin (DG) solvers on hybrid computers. The applications is electromagnetic simulations for the conception of antenna, electronic devices or aircraft electromagnetic compatibility.
- The acoustic conception of large rooms requires huge numerical simulations. It is not always possible to solve the full wave equation and many reduced acoustic models have been developed. A popular model consists in considering "acoustic" particles moving at the speed of sound. The resulting Partial Differential Equation (PDE) is very similar to the Vlasov equation. The same modeling is used in radiation theory. We have started to work on the reduction of the acoustic particles model and realized that our reduction approach perfectly applies to this situation. We plan to supervise a new PhD with CEREMA (Centre d'études et d'expertise sur les risques, l'environnement, la mobilité et l'aménagement) in Strasbourg. The objective is to investigate the model reduction and to implement the resulting acoustic model in our DG solver.

5. New Software and Platforms

5.1. SeLaLib

The objective of the Selalib project (SEmi-LAgrangian LIBrary) is to develop a well-designed, organized and documented library implementing several numerical methods for kinetic models of plasma physics. Its ultimate goal is to produce gyrokinetic simulations.

Another objective of the library is to provide to physicists easy-to-use gyrokinetic solvers, based on the semi-lagrangian techniques developed by Eric Sonnendrücker and his collaborators in the past CALVI project. The new models and schemes from TONUS are also intended to be incorporated into Selalib.

In addition, the CEA of Cadarache is interested by the development of this library, which picks up and extends many methods implemented in GYSELA, a code developed at CEA Cadarache for simulating turbulence in magnetic fusion plasmas, in particular, in view of the ITER project. Eric Sonnendrücker who is now in Munich continues to work on Selalib. A joint development of Selalib between Strasbourg and Munich allows both partners to benefit of each other's work.

Selalib is a library of FORTRAN modules. The CEA Cadarache has advised this language, because it is widespread in the engineering and physics communities. In this way, we hope that it will be spread among researchers interested in plasma simulations.

Selalib is under GPL license and available on the Inria Forge ⁴.

5.2. CLAC

CLAC is a generic Discontinuous Galerkin solver, written in C/C++, based on the OpenCL and MPI frameworks. CLAC means "Conservation Laws Approximation on many Cores".

It is clear now that future computers will be made of a collection of thousands of interconnected multicore processors. Globally it appears as a classical distributed memory MIMD machine. But at a lower level, each of the multicore processors is itself made of a shared memory MIMD unit (a few classical CPU cores) and a SIMD unit (a GPU). When designing new algorithms, it is important to adapt them to this kind of architecture. Our philosophy will be to program our algorithms in such a way that they can be run efficiently on this kind of computers. Practically, we will use the MPI library for managing the coarse grain parallelism, while the OpenCL library will efficiently operate the fine grain parallelism.

⁴<http://selalib.gforge.inria.fr/>

We have invested for several years until now into scientific computing on GPUs, using the open standard OpenCL (Open Computing Language). We were recently awarded a prize in the international AMD OpenCL innovation challenge thanks to an OpenCL two-dimensional Vlasov-Maxwell solver that fully runs on a GPU. OpenCL is a very interesting tool because it is an open standard now available on almost all brands of multicore processors and GPUs. The same parallel program can run on a GPU or a multicore processor without modification.

CLAC is also a joint project with a Strasbourg small company, AxesSim, which develops software for electromagnetic simulations.

Because of the envisaged applications of CLAC, which may be either academic or commercial, it is necessary to conceive a modular framework. The heart of the library is made of generic parallel algorithms for solving conservation laws. The parallelism can be both fine-grained (oriented towards GPUs and multicore processors) and coarse-grained (oriented towards GPU clusters). The separate modules allow managing the meshes and some specific applications. In this way, it is possible to isolate parts that should be protected for trade secret reasons. The open source part of CLAC will be made freely available on the web later on. We have made an APP deposit of the first version of CLAC in October 2012. The versioning of CLAC project is also registered in the Inria Forge ⁵.

6. New Results

6.1. Highlights of the Year

We have implemented an OpenCL task graph version of our Discontinuous Galerkin solver that allows to overlap GPU computations and MPI communications. With this optimizations, we were recently able to achieve a 14 GFLOPS simulation with 8 GPUs on an electromagnetic test case. These results are included in the PhD of Thomas Strub (defence planned in March 2015) under the supervision of Philippe Helluy.

6.2. Development of semi-Lagrangian methods

Participants: Adnane Hamiaz, Michel Mehrenberger, Christophe Steiner.

6.2.1. Gyroaverage operator for a polar mesh

A direct method is proposed in [17] in the space configuration for the computation of the gyroaverage operator. It consists in integrating on the gyrocircles using interpolation operators (Hermite or cubic splines); see also [2]. Numerical comparisons with a standard method based on a Padé approximation are performed: (i) with analytical solutions; (ii) considering the 4D drift-kinetic model with one Larmor radius and (iii) on the classical linear DIII-D benchmark case. In particular, we show that in the context of a drift-kinetic simulation, the proposed method has similar computational cost as the standard method and its precision is independent of the radius. Extension to the quasi neutral equation has begun on a 4D model with one Larmor radius. We can exhibit some specific situations where the new method leads to more accurate results and we observe as predicted that the instability growth rate is stronger than for the Padé approximation. On the other hand, we have to face with more oscillations (e.g. on the boundary) of the new operator, which does not permit to replace the Padé approximation. Promising higher order Padé approximation are envisaged for the future.

6.2.2. Semi-Lagrangian simulations on curvilinear grids

Semi-Lagrangian schemes often deal with cartesian mesh; the extension to curvilinear grids is important in order to be able to deal with specific geometries and also for adapting the grid to save computational effort. This study is part of a general work on adding curvilinear capabilities for the simulation of drift kinetic and gyrokinetic equations in a semi-Lagrangian framework, and is in current development in the SeLaLib library.

⁵<http://clac.gforge.inria.fr>

Thus, in [28] semi-Lagrangian guiding center simulations are performed on sinusoidal perturbations of cartesian grids, thanks to the use of a B-spline finite element solver for the Poisson equation and the classical backward semi-Lagrangian method (BSL) for the advection. We are able to reproduce the standard Kelvin-Helmholtz instability test on such grids. When the perturbation leads to a strong distorted mesh, we observe that the solution differs if one takes standard numerical parameters that are used in the cartesian reference case. We can recover good results together with correct mass conservation, by diminishing the time step.

6.2.3. *Field aligned semi-Lagrangian schemes*

In [23] we introduce field aligned interpolation for Semi-Lagrangian schemes, by adapting a method developed by Hariri-Ottaviani to the semi-Lagrangian context. This approach is validated on the constant oblique advection equation and on a 4D drift kinetic model with oblique magnetic field in cylindrical geometry. The strength of this method is that one can reduce the number of points in the longitudinal direction. Extension to tokamak configuration in toroidal geometry is the next step of this study.

6.2.4. *KEEN wave simulations, high order time splitting, non-uniform cubic splines*

KEEN waves are non-stationary, nonlinear, self-organized asymptotic states in Vlasov plasmas (see [3]). They lie outside the precepts of linear theory or perturbative analysis, unlike electron plasma waves or ion acoustic waves. Steady state, nonlinear constructs such as BGK modes also do not apply. The range in velocity that is strongly perturbed by KEEN waves depends on the amplitude and duration of the ponderomotive force generated by two crossing laser beams, for instance, used to drive them. Smaller amplitude drives manage to devolve into multiple highly-localized vorticelets, after the drive is turned off, and may eventually succeed to coalesce into KEEN waves. Fragmentation once the drive stops, and potential eventual remerger, is a hallmark of the weakly driven cases. A fully formed (more strongly driven) KEEN wave has one dominant vortical core. But it also involves fine scale complex dynamics due to shedding and merging of smaller vortical structures with the main one. Shedding and merging of vorticelets are involved in either case, but at different rates and with different relative importance. The narrow velocity range in which one must maintain sufficient resolution in the weakly driven cases, challenges fixed velocity grid numerical schemes. What is needed is the capability of resolving locally in velocity while maintaining a coarse grid outside the highly perturbed region of phase space. We here report on a new Semi-Lagrangian Vlasov-Poisson solver based on conservative non-uniform cubic splines in velocity that tackles this problem head on. An additional feature of our approach is the use of a new high-order time-splitting scheme which allows much longer simulations per computational effort. This is needed for low amplitude runs. There, global coherent structures take a long time to set up, such as KEEN waves, if they do so at all. The new code's performance is compared to uniform grid simulations and the advantages are quantified. The birth pains associated with weakly driven KEEN waves are captured in these simulations. Canonical KEEN waves with ample drive are also treated using these advanced techniques. They will allow the efficient simulation of KEEN waves in multiple dimensions, which will be tackled next, as well as generalizations to Vlasov-Maxwell codes. These are essential for pursuing the impact of KEEN waves in high energy density plasmas and in inertial confinement fusion applications. More generally, one needs a fully-adaptive grid in- phase-space method which could handle all small vorticelet dynamics whether peeling or remerging. Such fully adaptive grids would have to be computed sparsely in order to be viable. This two-velocity grid method is a concrete and fruitful step in that direction.

6.2.5. *Conservative semi-Lagrangian scheme*

While developing a new semi-Lagrangian solver, the gap between a linear Landau run in 1D-1D and a 5D gyrokinetic simulation in toroidal geometry is quite huge. Intermediate test cases are welcome for testing the code. A new fully two-dimensional conservative semi-Lagrangian (CSL) method is presented in [6] and is validated on 2D polar geometries. We consider here as building block, a 2D guiding-center type equation on an annulus and apply it on two test cases. First, we revisit a 2D test case previously done with a PIC approach and detail the boundary conditions. Second, we consider a 4D drift-kinetic slab simulation. In both cases, the new method appears to be a good alternative to deal with this type of models since it improves the lack of mass conservation of the standard semi-Lagrangian (BSL) method.

6.3. Reduced Vlasov-Maxwell modeling

Participants: Philippe Helluy, Laurent Navoret, Thi Trang Nhung Pham.

We have tested several preliminary methods for reducing the complexity of the Vlasov equation. By expanding the distribution function on velocity basis we obtain a space-only hyperbolic system. This system takes advantage of interesting conservation or entropy properties. Several types of basis can be used: Fourier [14], piecewise Lagrange [20], [13], etc. The method has been implemented for 4D problems in the Selalib library. The next step would be to adapt the size of the expansion according to the nature of the flow region and to apply the method to the gyrokinetic model.

6.4. GPU Optimization of Discontinuous Galerkin solvers

Participants: Michaël Gutnic, Philippe Helluy, Michel Massaro, Thomas Strub.

We have continued to investigate implementations of numerical schemes on new hybrid computer architectures. We have for instance applied a very efficient Strang splitting algorithm for the numerical resolution of the MHD or compressible multiphase model ([16], [18], [22]). We have also highly optimized our DG solver CLAC ([20]) for electromagnetic applications. For instance, we have implemented an OpenCL task graph that allows to overlap GPU computations and MPI communications. With this optimizations, we were recently able to achieve a 14 GFLOPS simulation with 8 GPUs on an electromagnetic test case. These results are included in the PhD of Thomas Strub (defence planned in March 2015).

6.5. Numerical and theoretical study of reduced MHD problems for the JOREK code

Participant: Emmanuel Franck.

The Jorek code is a parallel finite element code (used at the CEA Cadarache and the IPP) which simulates the edge instabilities in the Tokamak solving reduced MHD models. Firstly we have written a family of full MHD models (resistive, diamagnetic and extended MHD models). Using this, we write the reduced MHD models close to the models implemented in the code which conserve the energy and are more stable ([35]). This work will probably be published as an Inria report next year. The second part of this work consists in writing a simplified version of the JOREK code which will be useful to test and validate future numerical research in the JOREK context. Actually we have written a code which solve simple elliptic equations in 3D toroidal geometry using Bezier, splines and Fourier expansion. The integration of simple wave model and reduced MHD models [33] is in progress. When these model will be implemented, we will test a new preconditioning for the JOREK code in these simple configurations.

6.6. Simulations of highly oscillatory Vlasov-type models

Participants: Emmanuel Frénod [Univ. Bretagne-Sud], Sever Hirstoaga.

We continued our exploration of a new time-stepping method based on an exponential integrator.

First, we have improved the algorithm introduced in [11] for solving a multi-scale 1d-1d Vlasov-Poisson system within a Particle-In-Cell method, in order to do accurate long time simulations. As an exponential integrator, the new scheme (see [10]) allows to use large time steps compared to the size of oscillations in the solution. More precisely, the new idea is to push each particle with its computed period. Our simulations show that using precise periods for each particle and at each macroscopic time step results in a more accurate scheme in long times.

Then, similar ideas are used for a 2d-2d multi-scale Vlasov-Poisson system (see [27]). We propose in a Particle-In-Cell framework a robust time-stepping method that works uniformly when the small parameter (the smallest scale) vanishes. We first verify our scheme in the framework of a proposed analytic solution with fast oscillations in time and we show that the scheme works for any initial condition. Then we test the method in the nonlinear case of a Vlasov-Poisson simulation. The scheme is able to use large time steps with respect to the typical size of the solution's fast oscillations. In addition, we show numerically that the method has accurate long time behaviour and that it is asymptotic preserving with respect to the limiting Guiding Center system.

7. Bilateral Contracts and Grants with Industry

7.1. Bilateral Contracts with Industry

We are participating to a project with company AxesSim in Strasbourg. The objective is to help to the development of a commercial software for the numerical simulation of electromagnetic phenomena. The applications are directed towards antenna design and electromagnetic compatibility. This project is partly supported by DGA through "RAPID" (régime d'appui à l'innovation duale) funds. The CIFRE PhD of Thomas Strub is part of this project. Another CIFRE PhD will start in AxesSim on the same kind of subjects in March 2015.

8. Partnerships and Cooperations

8.1. National Initiatives

8.1.1. ANR

- ANR project "PEPPSI" in Programme Blanc SIMI 9 – Sciences de l'ingénierie (Edition 2012) started in 2013. Participants : Giovanni Manfredi (coordinator), Edwin Chacon Golcher, Sever Hirstoaga.

8.1.2. Euratom-CEA projects

- Michel Mehrenberger and Philippe Helluy are local coordinators of the project FR FCM (CNRS Federation on Magnetic Confinement Fusion), within Euratom-CEA association, Title:"Numerical Methods for GYSELA", the goal is to help improving the numerical algorithms used by the GYSELA code developed at CEA Cadarache for the simulation of turbulence in magnetic fusion plasmas.

8.2. European Initiatives

8.2.1. FP7 & H2020 Projects

The members of the team were in the following EUROfusion research projects:

E. Frénod, P. Helluy, S. Hirstoaga, M. Mehrenberger, L. Navoret were members of the project *CfP-WP14-ER-01/IPP-03*:

Max-Planck Institute for Plasma Physics, Garching (PI: Eric Sonnendrücker)

“Verification of global gyrokinetic codes and development of new algorithms for gyrokinetic and kinetic codes”

E. Frénod was member of the project *CfP-WP14-ER-01/Swiss Confederation-01*

École Polytechnique Fédérale de Lausanne (PI: Paolo Ricci)

“Synergetic numerical-experimental approach to fundamental aspects of turbulent transport in the tokamak edge”

E. Franck was member of the EUROfusion Enabling Research Project
CEA Cadarache, IRFM/SIPP/GP2B (PI: Marina Becoulet)
“JOREK, BOUT++ non-linear MHD modelling of MHD instabilities and their control in existing tokamaks and ITER”

8.3. International Research Visitors

8.3.1. Visits to International Teams

8.3.1.1. Research stays abroad

Michel Mehrenberger was on secondment at the Max Planck Institute in Munich until September 1st, 2014.

Emmanuel Frénod was invited professor during May 2014 at the Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai - China.

9. Dissemination

9.1. Promoting Scientific Activities

9.1.1. Scientific events selection

9.1.1.1. Chair of conference program committee

Emmanuel Frénod : AIMS Conference - Special Session “Homogenization Based Numerical Methods - Madrid - 7 - 11 juillet, 2014

9.1.2. Journal

9.1.2.1. Member of the editorial board

Emmanuel Frénod : Chinese Journal of Mathematics and Discrete and Continuous Dynamical Systems - Series S from July 2014.

Emmanuel Frénod : invited editor to a special issue in DCDS-S, « Numerical Methods Based on Two-Scale Convergence and Homogenization », to appear in 2015.

9.1.2.2. Reviewer

Michel Mehrenberger for the journals: Journal of Computational Physics, Journal of Graphics Tools, Communications in Computational Physics, Electronic Journal of Qualitative Theory of Differential Equations, Applied Mathematics and Computation, Computer Physics Communications, Journal of Computational and Applied Mathematics.

Philippe Helluy for the journals: maths reviews, SINUM, IJFV, Computers and Fluids, International Journal for Numerical Methods in Fluids, ESAIM Procs, PIER Journal, Journal of Mechanical Science and Technology,

9.2. Teaching - Supervision - Juries

9.2.1. Teaching

Licence :

Philippe Helluy: Calcul scientifique, 69 h eq. TD, L2, Université de Strasbourg, France.

Laurent Navoret, Calcul scientifique, 65 h eq. TD, L3, Université de Strasbourg, France.

Michel Mehrenberger, Analyse Mathématique d'une variable réelle, 87.5 h eq. TD, L1, Strasbourg, France.

Michel Mehrenberger, Outils mathématiques, 26.25h eq. TD, L3, Strasbourg France.

Michaël Gutnic, Mathématiques pour les sciences du vivant, 60 h eq. TD, L1 Sciences du Vivant, Université de Strasbourg, France.

Michaël Gutnic, Statistiques pour les biologistes, 117 h eq. TD, L2 Sciences du Vivant, Université de Strasbourg, France.

Master:

Philippe Helluy, Préparation à l'agrégation, 24 h eq. TD, M2 Agrégation, Université de Strasbourg, France.

Laurent Navoret, Calcul scientifique, 31 h eq. TD, M2 Agrégation, Université de Strasbourg, France.

Michel Mehrenberger, TP Calcul Scientifique Agrégation S3, 28h eq. TD, M2 Agrégation, Strasbourg France.

Ingénieur :

Michaël Gutnic, Probabilités et Statistiques, 30 h eq. TD, Formation d'ingénieur en informatique en apprentissage, Institut des Techniques d'Ingénieur de l'Industrie, Centre de Formation d'Apprentis de l'Industrie, Conservatoire national des arts et métiers, France.

9.2.2. Supervision

PhD in progress : Thi Trang Nhung Pham, Méthodes numériques pour Vlasov, October 2012, Université de Strasbourg, Advisors: Philippe Helluy, Laurent Navoret.

PhD in progress : Thomas Strub, Résolution des équations de Maxwell tridimensionnelles instationnaires sur architecture massivement multicœur, October 2011, Université de Strasbourg, Advisor: Philippe Helluy.

9.2.3. Juries

Nicolas Besse was referee in the PhD defense committee of Christophe Steiner.

Philippe Helluy and Michel Mehrenberger were in the PhD defense committee of Pierre Glanc and of Christophe Steiner.

Emmanuel Frénod was member of the HdR defense committee of Evans Gouno, January 31, 2014.

Philippe Helluy was referee or in the defense committee of: T. Volkert, J. Karel, M. Etancelin, Y. Dugout, M. Mounier.

9.3. Popularization

Philippe Helluy has given a conference at the "Maison des sciences" in Strasbourg. This event is proposed for the formation of middle and high school teachers. The title of the conference was "Modèles mathématiques pour la météo".

10. Bibliography

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- [2] C. STEINER. *Numerical methods for the gyroaverage operator, advection schemes and coupling. Applications to the Vlasov equation.*, Université de Strasbourg, December 2014, <https://tel.archives-ouvertes.fr/tel-01098173>

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