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ACTIVITY REPORT

Project-Team MOKAPLAN

Advances in Numerical Calculus of Variations

IN COLLABORATION WITH: CEREMADE

DOMAIN

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THEME

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Project-Team MOKAPLAN

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2 Overall objectives

2.1 Introduction

The last decade has witnessed a remarkable convergence between several sub-domains of the calculus of variations, namely optimal transport (and its many generalizations), infinite dimensional geometry of diffeomorphisms groups and inverse problems in imaging (in particular sparsity-based regularization). This convergence is due to (i) the mathematical objects manipulated in these problems, namely sparse measures (e.g. coupling in transport, edge location in imaging, displacement fields for diffeomorphisms) and (ii) the use of similar numerical tools from non-smooth optimization and geometric discretization schemes. Optimal Transportation, diffeomorphisms and sparsity-based methods are powerful modeling tools, that impact a rapidly expanding list of scientific applications and call for efficient numerical strategies. Our research program shows the important part played by the team members in the development of these numerical methods and their application to challenging problems.

2.2 Static Optimal Transport and Generalizations

Optimal Transport, Old and New. *Optimal Mass Transportation* is a mathematical research topic which started two centuries ago with Monge's work on the "Théorie des déblais et des remblais" (see [95]). This engineering problem consists in minimizing the transport cost between two given mass densities. In the 40's, Kantorovich [104] introduced a powerful linear relaxation and introduced its dual formulation. The *Monge-Kantorovich* problem became a specialized research topic in optimization and Kantorovich obtained the 1975 Nobel prize in economics for his contributions to resource allocations problems. Since the seminal discoveries of Brenier in the 90's [57], Optimal Transportation has received renewed attention from mathematical analysts and the Fields Medal awarded in 2010 to C. Villani, who gave important contributions to Optimal Transportation and wrote the modern reference monographs [129, 130], arrived at a culminating moment for this theory. Optimal Mass Transportation is today a mature area of mathematical analysis with a constantly growing range of applications. Optimal Transportation has also received a lot of attention from probabilists (see for instance the recent survey [108] for an overview of the Schrödinger problem which is a stochastic variant of the Benamou-Brenier dynamical formulation of optimal transport). The development of numerical methods for Optimal Transportation and Optimal Transportation related problems is a difficult topic and comparatively underdeveloped. This research field has experienced a surge of activity in the last five years, with important contributions of the MOKAPLAN group (see the list of important publications of the team). We describe below a few of recent and less recent Optimal Transportation concepts and methods which are connected to the future activities of MOKAPLAN :

Brenier's theorem [59] characterizes the unique optimal map as the gradient of a convex potential. As such Optimal Transportation may be interpreted as an infinite dimensional optimisation problem under "convexity constraint": i.e. the solution of this infinite dimensional optimisation problem is a convex potential. This connects Optimal Transportation to "convexity constrained" non-linear variational problems such as, for instance, Newton's problem of the body of minimal resistance. The value function of the optimal transport problem is also known to define a distance between source and target densities called the *Wasserstein distance* which plays a key role in many applications such as image processing.

Monge-Ampère Methods. A formal substitution of the optimal transport map as the gradient of a convex potential in the mass conservation constraint (a Jacobian equation) gives a non-linear Monge-Ampère equation. Caffarelli [63] used this result to extend the regularity theory for the Monge-Ampère equation. In the last ten years, it also motivated new research on numerical solvers for non-linear degenerate Elliptic equations [87] [112] [49] [50] and the references therein. Geometric approaches based on Laguerre diagrams and discrete data [115] have also been developed. Monge-Ampère based Optimal Transportation solvers have recently given the first linear cost computations of Optimal Transportation (smooth) maps.

Generalizations of OT. In recent years, the classical Optimal Transportation problem has been extended in several directions. First, different ground costs measuring the "physical" displacement have been considered. In particular, well posedness for a large class of convex and concave costs has been established by McCann and Gangbo [94]. Optimal Transportation techniques have been applied for example to a Coulomb ground cost in Quantum chemistry in relation with Density Functional theory [81]. Given the densities of electrons

Figure 1: Example of color transfer between two images, computed using the method developed in [47], see also [124]. The image framed in red and blue are the input images. *Top and middle row:* adjusted image where the color of the transported histogram has been imposed. *Bottom row:* geodesic (displacement) interpolation between the histogram of the chrominance of the image.

Optimal Transportation models the potential energy and their relative positions. For more than more than 2 electrons (and therefore more than 2 densities) the natural extension of Optimal Transportation is the so called Multi-marginal Optimal Transport (see [119] and the references therein). Another instance of multi-marginal Optimal Transportation arises in the so-called Wasserstein barycenter problem between an arbitrary number of densities [34]. An interesting overview of this emerging new field of optimal transport and its applications can be found in the recent survey of Ghoussoub and Pass [120].

Numerical Applications of Optimal Transportation. Optimal transport has found many applications, starting from its relation with several physical models such as the semi-geostrophic equations in meteorology [99, 84, 83, 45, 111], mesh adaptation [110], the reconstruction of the early mass distribution of the Universe [92, 60] in Astrophysics, and the numerical optimisation of reflectors following the Optimal Transportation interpretation of Oliker [64] and Wang [131]. Extensions of OT such as multi-marginal transport has potential applications in Density Functional Theory , Generalized solution of Euler equations [58] (DFT) and in statistics and finance [43, 93] .... Recently, there has been a spread of interest in applications of OT methods in imaging sciences [53], statistics [51] and machine learning [85]. This is largely due to the emergence of fast numerical schemes to approximate the transportation distance and its generalizations, see for instance [47]. Figure 1 shows an example of application of OT to color transfer. Figure 2 shows an example of application in computer graphics to interpolate between input shapes.

2.3 Diffeomorphisms and Dynamical Transport

Dynamical transport. While the optimal transport problem, in its original formulation, is a static problem (no time evolution is considered), it makes sense in many applications to rather consider time evolution. This is relevant for instance in applications to fluid dynamics or in medical images to perform registration of organs and model tumor growth.

In this perspective, the optimal transport in Euclidean space corresponds to an evolution where each particule of mass evolves in straight line. This interpretation corresponds to the *Computational Fluid Dynamic* (CFD) formulation proposed by Brenier and Benamou in [44]. These solutions are time curves in the space of densities and geodesics for the Wasserstein distance. The CFD formulation relaxes the non-linear mass conservation constraint into a time dependent continuity equation, the cost function remains convex but is highly non smooth. A remarkable feature of this dynamical formulation is that it can be re-cast as a convex but non smooth optimization problem. This convex dynamical formulation finds many non-trivial extensions and applications, see for instance [46]. The CFD formulation also appears to be a limit case of *Mean Fields games*

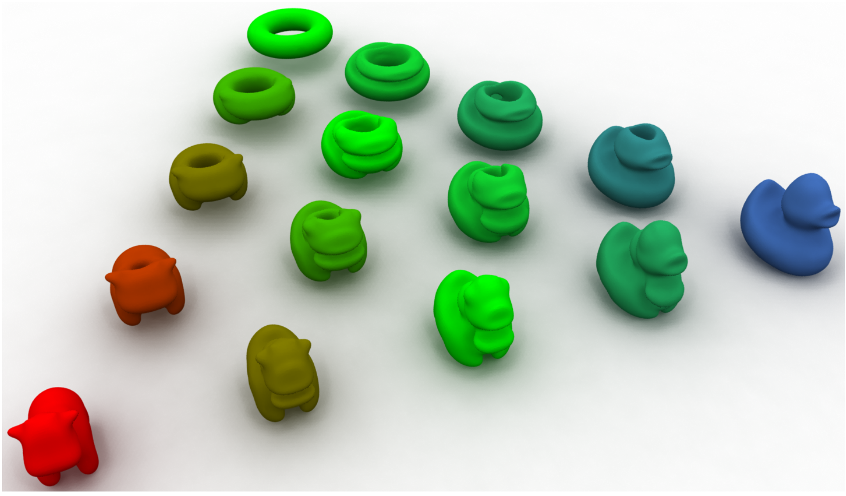

Figure 2: Example of barycenter between shapes computed using optimal transport barycenters of the uniform densities inside the 3 extremal shapes, computed as detailed in [124]. Note that the barycenters are not in general uniform distributions, and we display them as the surface defined by a suitable level-set of the density.

(MFGs), a large class of economic models introduced by Lasry and Lions [106] leading to a system coupling an Hamilton-Jacobi with a Fokker-Planck equation. In contrast, the Monge case where the ground cost is the euclidan distance leads to a static system of PDEs [54].

Gradient Flows for the Wasserstein Distance. Another extension is, instead of considering geodesic for transportation metric (i.e. minimizing the Wasserstein distance to a target measure), to make the density evolve in order to minimize some functional. Computing the steepest descent direction with respect to the Wasserstein distance defines a so-called Wasserstein gradient flow, also known as *JKO gradient flows* after its authors [103]. This is a popular tool to study a large class of non-linear diffusion equations. Two interesting examples are the Keller-Segel system for chemotaxis [102, 73] and a model of congested crowd motion proposed by Maury, Santambrogio and Roudneff-Chupin [114]. From the numerical point of view, these schemes are understood to be the natural analogue of implicit scheme for linear parabolic equations. The resolution is however costly as it involves taking the derivative in the Wasserstein sense of the relevant energy, which in turn requires the resolution of a large scale convex but non-smooth minimization.

Geodesic on infinite dimensional Riemannian spaces. To tackle more complicated warping problems, such as those encountered in medical image analysis, one unfortunately has to drop the convexity of the functional involved in defining the gradient flow. This gradient flow can either be understood as defining a geodesic on the (infinite dimensional) group of diffeomorphisms [42], or on a (infinite dimensional) space of curves or surfaces [132]. The de-facto standard to define, analyze and compute these geodesics is the "Large Deformation Diffeomorphic Metric Mapping" (LDDMM) framework of Trouvé, Younes, Holm and co-authors [42, 98]. While in the CFD formulation of optimal transport, the metric on infinitesimal deformations is just the L^2 norm (measure according to the density being transported), in LDDMM, one needs to use a stronger regularizing metric, such as Sobolev-like norms or reproducing kernel Hilbert spaces (RKHS). This enables a control over the smoothness of the deformation which is crucial for many applications. The price to pay is the need to solve a non-convex optimization problem through geodesic shooting method [116], which requires to integrate backward and forward the geodesic ODE. The resulting strong Riemannian geodesic structure on spaces of diffeomorphisms or shapes is also pivotal to allow us to perform statistical analysis on the tangent space, to define mean shapes and perform dimensionality reduction when analyzing large collection of input shapes (e.g. to study evolution of a diseases in time or the variation across patients) [65].

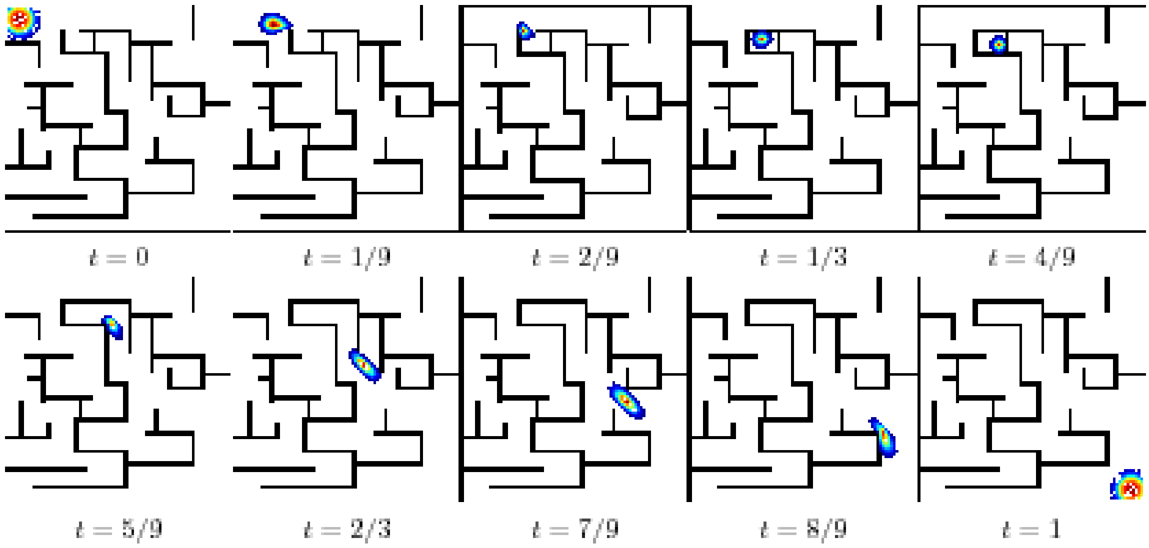

Figure 3: Examples of displacement interpolation (geodesic for optimal transport) according to a non-Euclidean Riemannian metric (the mass is constrained to move inside a maze) between to input Gaussian distributions. Note that the maze is dynamic: its topology change over time, the mass being “trapped” at time $t = 1/3$.

2.4 Sparsity in Imaging

Sparse ℓ^1 regularization. Beside image warping and registration in medical image analysis, a key problem in nearly all imaging applications is the reconstruction of high quality data from low resolution observations. This field, commonly referred to as “inverse problems”, is very often concerned with the precise location of features such as point sources (modeled as Dirac masses) or sharp contours of objects (modeled as gradients being Dirac masses along curves). The underlying intuition behind these ideas is the so-called sparsity model (either of the data itself, its gradient, or other more complicated representations such as wavelets, curvelets, bandlets [113] and learned representation [133]).

The huge interest in these ideas started mostly from the introduction of convex methods to serve as proxy for these sparse regularizations. The most well known is the ℓ^1 norm introduced independently in imaging by Donoho and co-workers under the name “Basis Pursuit” [78] and in statistics by Tibshirani [125] under the name “Lasso”. A more recent resurgence of this interest dates back to 10 years ago with the introduction of the so-called “compressed sensing” acquisition techniques [66], which make use of randomized forward operators and ℓ^1-type reconstruction.

Regularization over measure spaces. However, the theoretical analysis of sparse reconstructions involving real-life acquisition operators (such as those found in seismic imaging, neuro-imaging, astro-physical imaging, etc.) is still mostly an open problem. A recent research direction, triggered by a paper of Candès and Fernandez-Granda [68], is to study directly the infinite dimensional problem of reconstruction of sparse measures (i.e. sum of Dirac masses) using the total variation of measures (not to be mistaken for the total variation of 2-D functions). Several works [67, 89, 88] have used this framework to provide theoretical performance guarantees by basically studying how the distance between neighboring spikes impacts noise stability.

Low complexity regularization and partial smoothness. In image processing, one of the most popular methods is the total variation regularization [123, 61]. It favors low-complexity images that are piecewise constant, see Figure 4 for some examples on how to solve some image processing problems. Beside applications in image processing, sparsity-related ideas also had a deep impact in statistics [125] and machine learning [37]. As a typical example, for applications to recommendation systems, it makes sense to consider sparsity of the singular values of matrices, which can be relaxed using the so-called nuclear norm (a.k.a. trace norm) [38]. The underlying methodology is to make use of low-complexity regularization models, which turns out to be

Segmentation input — output — Zooming input — output

Figure 4: Two example of application of the total variation regularization of functions. *Left:* image segmentation into homogeneous color regions. *Right:* image zooming (increasing the number of pixels while keeping the edges sharp).

equivalent to the use of partly-smooth regularization functionals [109, 128] enforcing the solution to belong to a low-dimensional manifold.

2.5 MOKAPLAN unified point of view

The dynamical formulation of optimal transport creates a link between optimal transport and geodesics on diffeomorphisms groups. This formal link has at least two strong implications that MOKAPLAN will elaborate on: (i) the development of novel models that bridge the gap between these two fields ; (ii) the introduction of novel fast numerical solvers based on ideas from both non-smooth optimization techniques and Bregman metrics.

In a similar line of ideas, we believe a unified approach is needed to tackle both sparse regularization in imaging and various generalized OT problems. Both require to solve related non-smooth and large scale optimization problems. Ideas from proximal optimization has proved crucial to address problems in both fields (see for instance [44, 122]). Transportation metrics are also the correct way to compare and regularize variational problems that arise in image processing (see for instance the Radon inversion method proposed in [47]) and machine learning (see [85]).

3 Research program

Since its creation, the Mokaplan team has made important contributions in Optimal Transport both on the theoretical and the numerical side, together with applications such as fluid mechanics, the simulation biological systems, machine learning. We have also contributed to to the field of inverse problems in signal and image processing (super-resolution, nonconvex low rank matrix recovery). In 2022, the team was renewed with the following research program which broadens our spectrum and addresses exciting new problems.

3.1 OT and related variational problems solvers *encore et toujours*

Participants: Flavien Léger , Jean-David Benamou , Guillaume Carlier , Thomas Gallouët , Guillaume Chazareix , Adrien Vacher , Paul Pegon.

Asymptotic analysis of entropic OT for a small entropic parameter is well understood for regular data on compact manifolds and standard quadratic ground cost [79], the team will extend this study to more general settings and also establish rigorous asymptotic estimates for the transports maps. This is important to provide a sound theoretical background to efficient and useful debiasing approaches like Sinkhorn Divergences [90]. Guillaume Carlier, Paul Pegon and Luca Tamanini are investigating speed of convergence and quantitative stability results under general conditions on the cost (so that optimal maps may not be continuous or even fail to exist). Some sharp bounds have already been obtained, the next challenging goal is to extend the Laplace method to a nonsmooth setting and understand what entropic OT really selects when there are several optimal OT plans.

High dimensional - Curse of dimensionality We will continue to investigate the computation or approximation of high-dimensional OT losses and the associated transports [127] in particular in relation with their use in ML. In particular for Wasserstein 2 metric but also the repulsive Density Functional theory cost [91].

Back-and-forth The back-and-forth method [101, 100] is a state-of-the-art solver to compute optimal transport with convex costs and 2-Wasserstein gradient flows on grids. Based on simple but new ideas it has great potential to be useful for related problems. We plan to investigate: OT on point clouds in low dimension, the principal-agent problem in economics and more generally optimization under convex constraints [105, 117].

Transport and diffusion The diffusion induced by the entropic regularization is fixed and now well understood. For recent variations of the OT problem (Martingale OT, Weak OT see [39]) the diffusion becomes an explicit constraint or the control itself [96]. The entropic regularisation of these problems can then be understood as metric/ground cost learning [69] (see also [121]) and offers a tractable numerical method.

Wasserstein Hamiltonian systems We started to investigate the use of modern OT solvers for the SG equation [82, 45] Semi-Discrete and entropic regularization. This is a special instance Hamiltonian Systems in the sense of [35]. with an OT component in the Energy.

Nonlinear fourth-order diffusion equations such as thin-films or (the more involved) DLSS quantum drift equations are WGF. Such WGF are challenging both in terms of mathematical analysis (lack of maximum principle...) and of numerics. They are currently investigated by Jean-David Benamou, Guillaume Carlier in collaboration with Daniel Matthes. Note also that Mokaplan already contributed to a related topic through the TV-JKO scheme [71].

Lagrangian approaches for fluid mechanics More generally we want to extend the design and implementation of Lagrangian numerical scheme for a large class of problem coming from fluids mechanics (WHS or WGF) using semi-discrete OT or entropic regularization. We will also take a special attention to link this approaches with problems in machine/statistical learning. To achieve this part of the project we will join forces with colleagues in Orsay University: Y. Brenier, H. Leclerc, Q. Mérigot, L. Nenna.

L^∞ **optimal transport** is a variant of OT where we want to minimize the maximal displacement of the transport plan, instead of the average distance. Following the seminal work of [77], and more recent developments [86], Guillaume Carlier, Paul Pegon and Luigi De Pascale are working on the description of *restrictable* solutions (which are cyclically ∞-monotone) through some potential maps, in the spirit of Mange-Kantorovich potentials provided by a duality theory. Some progress has been made to partially describe cyclically quasi-motonone maps (related in some sense to cyclically ∞-monotone maps), through quasi-convex potentials.

3.2 Application of OT numerics to non-variational and non convex problems

Participants: Flavien Léger , Guillaume Carlier , Jean-David Benamou.

Market design Z-mappings form a theory of non-variational problems initiated in the '70s but that has been for the most part overlooked by mathematicians. We are developing a new theory of the algorithms associated with convergent regular splitting of Z-mappings. Various well-established algorithms for matching models can be grouped under this point of view (Sinkhorn, Gale–Shapley, Bertsekas' auction) and this new perspective has the potential to unlock new convergence results, rates and accelerated methods.

Non Convex inverse problems The PhD [134] provided a first exploration of Unbalanced Sinkhorn Divergence in this context. Given enough resources, a branch of PySit, a public domain software to test misfit functions in the context of Seismic imaging will be created and will allow to test other signal processing strategies in Full Waveform Inversion. Likewise the numerical method tested for 1D reflectors in [48] coule be develloped further (in particular in 2D).

Equilibrium and transport Equilibrium in labor markets can often be expressed in terms of the Kantorovich duality. In the context of urban modelling or spatial pricing, this observation can be fruitfully used to compute equilibrium prices or densities as fixed points of operators involving OT, this was used in [41] and [40]. Quentin Petit, Guillaume Carlier and Yves Achdou are currently developing a (non-variational) new semi-discrete model for the structure of cities with applications to tele-working.

Non-convex Principal-Agent problems Guillaume Carlier, Xavier Dupuis, Jean-Charles Rochet and John Thanassoulis are developing a new saddle-point approach to non-convex multidimensional screening problems arising in regulation (Barron-Myerson) and taxation (Mirrlees).

3.3 Inverse problems with structured priors

Participants: Irène Waldspurger , Antonin Chambolle , Vincent Duval , Faniriana Rakoto Endor , Annette Dumas.

Off-the-grid reconstruction of complex objects Whereas, very recently, some methods were proposed for the reconstruction of curves and piecewise constant images on a continuous domain ([56] and [72]), those are mostly proofs of concept, and there is still some work to make them competitive in real applications. As they are much more complex than point source reconstruction methods, there is room for improvements (parametrization, introduction of several atoms...). In particular, we are currently working on an improvement of the algorithm [56] for inverse problems in imaging which involve Optimal Transport as a regularizer (see [126] for preliminary results). Moreover, we need to better understand their convergence and the robustness of such methods, using sensitivity analysis.

Correctness guarantees for Burer-Monteiro methods Burer-Monteiro methods work well in practice and are therefore widely used, but existing correctness guarantees [55] hold under unrealistic assumptions only. In the long term, we aim at proposing new guarantees, which would be slightly weaker but would hold in settings more relevant to practice. A first step is to understand the “average” behavior of Burer-Monteiro methods, when applied to random problems, and could be the subject of a PhD thesis.

3.4 Geometric variational problems, and their interactions with transport

Participants: Vincent Duval , Paul Pegon , Antonin Chambolle , Joao-Miguel Machado .

Approximation of measures with geometric constraints Optimal Transport is a powerful tool to compare and approximate densities, but its interaction with geometric constraints is still not well understood. In applications such as optimal design of structures, one aims at approximating an optimal pattern while taking into account fabrication constraints [52]. In Magnetic Resonance Imaging (MRI), one tries to sample the Fourier transform of the unknown image according to an optimal density but the acquisition device can only proceed along curves with bounded speed and bounded curvature [107]. Our goal is to understand how OT interacts with energy terms which involve, e.g. the length, the perimeter or the curvature of the support... We want to understand the regularity of the solutions and to quantify the approximation error. Moreover, we want to design numerical methods for the resolution of such problems, with guaranteed performance.

Discretization of singular measures Beyond the (B)Lasso and the total variation (possibly off-the-grid), numerically solving branched transportation problems requires the ability to faithfully discretize and represent 1-dimensional structures in the space. The research program of A. Chambolle consists in part in developing the numerical analysis of variational problems involving singular measures, such as lower-dimensional currents or free surfaces. We will explore both phase-field methods (with P. Pegon, V. Duval) [74, 118] which easily represent non-convex problems, but lack precision, and (with V. Duval) precise discretizations of convex problems, based either on finite elements (and relying to the FEM discrete exterior calculus [36], cf [75] for the case of the total variation), or on finite differences and possibly a clever design of dual constraints as studied in [80, 76] again for the total variation.

Transport problems with metric optimization In urban planning models, one looks at building a network (of roads, metro or train lines, etc.) so as to minimize a transport cost between two distributions, penalized by the cost for building the network, usually its length. A typical transport cost is Monge cost MK_ω with a metric $\omega = \omega_\Sigma$ which is modified as a fraction of the euclidean metric on the network Σ. We would like to consider general problems involving a construction cost to generate a conductance field σ (having in mind 1-dimensional integral of some function of σ), and a transport cost depending on this conductance field. The afore-mentioned case studied in [62] falls into this category, as well as classical branched transport. The biologically-inspired network evolution model of [97] seems to provide such an energy in the vanishing diffusivity limit, with a cost for building a 1-dimensional permeability tensor and an L^2 congested transport cost with associated resistivity metric ; such a cost seems particularly relevant to model urban planning. Finally, we would like to design numerical methods to solve such problems, taking advantage of the separable structure of the whole cost.

4 Application domains

4.1 Natural Sciences

FreeForm Optics, Fluid Mechanics (Incompressible Euler, Semi-Geostrophic equations), Quantum Chemistry (Density Functional Theory), Statistical Physics (Schroedinger problem), Porous Media.

4.2 Signal Processing and inverse problems

Full Waveform Inversion (Geophysics), Super-resolution microscopy (Biology), Satellite imaging (Meteorology)

4.3 Social Sciences

Mean-field games, spatial economics, principal-agent models, taxation, nonlinear pricing.

5 Highlights of the year

We organized a workshop for the 10th anniversary of Mokaplan (MOKA10) at Ecole des Mines (Fontainebleau) on June, 4th to 6th. It gathered approximately 45 people who are close collaborators or former members of the team. It was a huge success.

5.1 Awards

Frontiers of Science 2024 prize (Beijing)for: "Iterative Bregman projections for regularized transportation problems", SIAM Journal on Scientific Computing (2015) Authors: Jean-David Benamou, Guillaume Carlier, Marco Cuturi, Luca Nenna, Gabriel Peyré

6 New results

6.1 From entropic transport to martingale transport, and applications to model calibration

Participants: Jean-David Benamou, Guillaume Chazareix, Gregoire Loeper.

[24]

We propose a discrete time formulation of the semi martingale optimal transport problem based on multi-marginal entropic transport. This approach offers a new way to formulate and solve numerically the calibration problem proposed by Guo et al. 2022, using a multi-marginal extension of Sinkhorn algorithm as in Benamou, Carlier, and Nenna 2019; Carlier et al. 2017; Benamou et al. 2019. In the limit when the time step goes to zero we recover, as detailed in the companion paper Benamou et al. 2024, a semi-martingale process, solution to a semi-martingale optimal transport problem, with a cost function involving the so-called specific entropy introduced in Gantert 1991, see also Föllmer 2022 and Backhoff-Veraguas and Unterberger 2023.

6.2 Entropic Semi-Martingale Optimal Transport

Participants: Jean-David Benamou, Guillaume Chazareix, Marc Hoffman, Gregoire Loeper, François-Xavier Vialard.

[23] Entropic Optimal Transport (EOT), also known as the Schrodinger problem, aims to find random processes with given initial and final marginals, minimizing the relative entropy (RE) with respect to a reference measure. Both processes (the reference and the controlled one) necessarily share the same diffusion coeffcients to ensure finiteness of the RE. This initially suggests that controlled-diffusion Semi-Martingale Optimal Transport (SMOT) problems may be incompatible with entropic regularization. However, when the process is observed at discrete times, forming a Markov chain, the RE remains finite even with variable diffusion coefficients. In this case, discrete semi-martingales can emerge as solutions to multi-marginal EOT problems. For smooth semi-martingales, the scaled limit of the relative entropy of their time discretizations converges to the specific relative entropy", a convex functional of the variance. This observation leads to an entropy regularized time discretization of the continuous SMOT problems, enabling the computation of discrete approximations via a multi-marginal Sinkhorn algorithm. We prove convergence of the time-discrete entropic problem to the continuous case, provide an implementation, and present numerical experiments supporting the theoretical results.

6.3 A geometric Laplace method

Participants: Flavien Léger, François-Xavier Vialard.

A classical tool for approximating integrals is the Laplace method. The first-order, as well as the higher-order Laplace formula is most often written in coordinates without any geometrical interpretation. In [9], motivated by a situation arising, among others, in optimal transport, we give a geometric formulation of the first-order term of the Laplace method. The central tool is the Kim–McCann Riemannian metric which was introduced in the field of optimal transportation. Our main result expresses the first-order term with standard geometric objects such as volume forms, Laplacians, covariant derivatives and scalar curvatures of two different metrics arising naturally in the Kim–McCann framework. Passing by, we give an explicitly quantified version of the Laplace formula, as well as examples of applications.

6.4 Convergence rate of general entropic optimal transport costs

Participants: Luca Nenna, Paul Pegon.

In [20] we improved and extended the results of [70] to the multi-marginal setting, investigating the convergence rate of the multi-marginal optimal entropic cost MOT_ε to the multi-marginal optimal transport cost as the noise parameter $\varepsilon \to 0$. We establish lower and upper bounds on the difference with the unregularized cost that depends on the dimensions of the marginals and on the ground cost, but not on the optimal transport plans themselves. In particular, we establish lower bounds for C^2 costs defined on the product of M submanifolds satisfying some signature condition on the mixed second derivatives that may include degenerate costs. We obtain in particular matching bounds in some typical situations where the optimal plan is deterministic, including the case of Wasserstein barycenters.

6.5 A geometric approach to apriori estimates for optimal transport maps

Participants: Simon Brendle, Flavien Leger, Robert J. McCann, Cale Rankin.

A key inequality which underpins the regularity theory of optimal transport for costs satisfying the Ma–Trudinger–Wang condition is the Pogorelov second derivative bound. This translates to an apriori interior C1 estimate for smooth optimal maps. Here we give a new derivation of this estimate which relies in part on Kim, McCann and Warren's observation that the graph of an optimal map becomes a volume maximizing spacelike submanifold when the product of the source and target domains is endowed with a suitable pseudo-Riemannian geometry that combines both the marginal densities and the cost.

6.6 Gradient descent with a general cost

Participants: Flavien Leger, Pierre-Cyril Aubin.

We present a new class of gradient-type optimization methods that extends vanilla gradient descent, mirror descent, Riemannian gradient descent, and natural gradient descent. Our approach involves constructing a surrogate for the objective function in a systematic manner, based on a chosen cost function. This surrogate is then minimized using an alternating minimization scheme. Using optimal transport theory we establish convergence rates based on generalized notions of smoothness and convexity. We provide local versions of these two notions when the cost satisfies a condition known as nonnegative cross-curvature. In particular our framework provides the first global rates for natural gradient descent and the standard Newton's method.

6.7 Variational approximation of H-masses

Participants: Paul Pegon, Antonin Monteil.

In [18] we consider first order local minimization problems of the form $\min \int_{\mathbb{R}^N} f(u, \nabla u)$ under a mass constraint $\int_{\mathbb{R}^N} u = m \in \mathbb{R}$. We prove that the minimal energy function $H(m)$ is always concave on $(-\infty, 0)$ and $(0, +\infty)$, and that relevant rescalings of the energy, depending on a small parameter ε, Γ-converge in the weak topology of measures towards the H-mass, defined for atomic measures $\sum_i m_i \delta_{x_i}$ as $\sum_i H(m_i)$. We also consider space inhomogeneous Lagrangians $f(x, u, \nabla u)$, which cover cases of space-inhomogeneous H-masses $\sum_i H(x_i, m_i)$, and also the case of a family of Lagrangians $(f_\varepsilon)_\varepsilon$ converging as $\varepsilon \to 0$. The Γ-convergence result holds under mild assumptions on f, and covers several situations including homogeneous H-masses in any dimension $N \geq 2$ for exponents above a critical threshold, and all concave H-masses in dimension $N = 1$. Our result yields in particular the concentration of Cahn-Hilliard fluids into droplets, and is related to the approximation of branched transport by elliptic energies.

6.8 From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows

Participants: Thomas Gallouët, Andrea Natale, Gabriele Todeschi.

We introduce a time discretization for Wasserstein gradient flows based on the classical Backward Differentiation Formula of order two. The main building block of the scheme is the notion of geodesic extrapolation in the Wasserstein space, which in general is not uniquely defined. We propose several possible definitions for such an operation, and we prove convergence of the resulting scheme to the limit PDE, in the case of the Fokker-Planck equation. For a specific choice of extrapolation we also prove a more general result, that is convergence towards EVI flows. Finally, we propose a variational finite volume discretization of the scheme which numerically achieves second order accuracy in both space and time.

6.9 Entropic approximation of infinity optimal transport problems

Participants: Camilla Brizzi, Guillaume Carlier , Luigi De-Pascale.

We propose an entropic approximation approach for optimal transportation problems with a supremal cost. We establish Γ-convergence for suitably chosen parameters for the entropic penalization and that this procedure selects ∞ cyclically monotone plans at the limit. We also present some numerical illustrations performed with Sinkhorn's algorithm.

6.10 Quantitative Stability of the Pushforward Operation by an Optimal Transport Map

Participants: Guillaume Carlier , Alex Delalande, Quentin Mérigot.

We study the quantitative stability of the mapping that to a measure associates its pushforward measure by a fixed (non-smooth) optimal transport map. We exhibit a tight Hölder-behavior for this operation under minimal assumptions. Our proof essentially relies on a new bound that quantifies the size of the singular sets of a convex and Lipschitz continuous function on a bounded domain.

6.11 Quantitative Stability of Barycenters in the Wasserstein Space

Participants: Guillaume Carlier , Alex Delalande, Quentin Mérigot.

Wasserstein barycenters define averages of probability measures in a geometrically meaningful way. Their use is increasingly popular in applied fields, such as image, geometry or language processing. In these fields however, the probability measures of interest are often not accessible in their entirety and the practitioner may have to deal with statistical or computational approximations instead. In this article, we quantify the effect of such approximations on the corresponding barycenters. We show that Wasserstein barycenters depend in a Hölder-continuous way on their marginals under relatively mild assumptions. Our proof relies on recent estimates that quantify the strong convexity of the dual quadratic optimal transport problem and a new result that allows to control the modulus of continuity of the push-forward operation under a (not necessarily smooth) optimal transport map.

6.12 Wasserstein medians: robustness, PDE characterization and numerics

Participants: Guillaume Carlier , Enis Chenchene, Katharina Eichinger.

We investigate the notion of Wasserstein median as an alternative to the Wasserstein barycenter, which has become popular but may be sensitive to outliers. In terms of robustness to corrupted data, we indeed show that Wasserstein medians have a breakdown point of approximately 1 2. We give explicit constructions of Wasserstein medians in dimension one which enable us to obtain L^p estimates (which do not hold in higher dimensions). We also address dual and multimarginal reformulations. In convex subsets of $\mathbb{R}^d$, we connect Wasserstein medians to a minimal (multi) flow problem à la Beckmann and a system of PDEs of Monge-Kantorovich-type, for which we propose a p-Laplacian approximation. Our analysis eventually leads to a new numerical method to compute Wasserstein medians, which is based on a Douglas-Rachford scheme applied to the minimal flow formulation of the problem.

6.13 Projected gradient descent accumulates at Bouligand stationary points

Participants: Guillaume Olikier, Irène Waldspurger.

In [33], we consider the problem of minimizing a continuously differentiable function on a nonempty closed subset of a Euclidean vector space using projected gradient descent (PGD). Without further assumptions on the objective function, PGD (as all standard algorithms) is not expected to find a global, or even local minimizer. Instead, it finds a so-called *stationary point.* Up to now, it was known that the accumulation points of sequences generated by PGD were *Mordukhovich stationary*, which is a relatively weak notion of stationarity. In the article, the accumulation points are proven to satisfy the stronger definition of *Bouligand stationarity*, and even *proximal stationarity* if the gradient is locally Lipschitz continuous. These are the strongest stationarity properties that can be expected for the considered problem.

6.14 Benign landscape for Burer-Monteiro factorizations of MaxCut-type semidefinite programs

Participants: Faniriana Rakoto Endor, Irène Waldspurger.

We study the Burer-Monteiro factorization, which is a well-known heuristic to reduce the computational cost of solving semidefinite programs (SDP) in the case where the solution is a priori known to be low rank. This factorization reduces the dimension of the SDP at the cost of its convexity, therefore possibly introducing spurious second-order critical points which could trap the optimization algorithm and prevent it from finding the desired minimizer.

In [31], for MaxCut-type SDP, we give a sharp condition on the conditioning of the associated Laplacian matrix under which any second-order critical point of the non-convex problem is a global minimizer. By applying our theorem, we improve on recent results about the correctness of the Burer-Monteiro factorization on $\mathbb{Z}^2$ -synchronization problems.

6.15 Nonnegative cross-curvature in infinite dimensions: synthetic definition and spaces of measures

Participants: Flavien Léger , Gabriele Todeschi, François-Xavier Vialard.

Nonnegative cross-curvature (NNCC) is a geometric property of a cost function defined on a product space that originates in optimal transportation and the Ma-Trudinger-Wang theory. Motivated by applications in optimization, gradient flows and mechanism design, we propose a variational formulation of nonnegative cross-curvature on c-convex domains applicable to infinite dimensions and nonsmooth settings. The resulting class of NNCC spaces is closed under Gromov-Hausdorff convergence and for this class, we extend many properties of classical nonnegative cross-curvature: stability under generalized Riemannian submersions, characterization in terms of the convexity of certain sets of c-concave functions, and in the metric case, it is a subclass of positively curved spaces in the sense of Alexandrov. One of our main results is that Wasserstein spaces of probability measures inherit the NNCC property from their base space. Additional examples of NNCC costs include the Bures-Wasserstein and Fisher-Rao squared distances, the Hellinger-Kantorovich squared distance (in some cases), the relative entropy on probability measures, and the 2-Gromov-Wasserstein squared distance on metric measure spaces.

6.16 Dynamical programming for off-the-grid dynamic inverse problems

Participants: Vincent Duval, Robert Tovey.

In [14], we consider off-the-grid algorithms for the reconstruction of sparse measures from time-varying data. In particular, the reconstruction is a finite collection of Dirac measures whose locations and masses vary continuously in time. Recent work showed that this decomposition was possible by minimising a convex variational model which combined a quadratic data fidelity with dynamical Optimal Transport. We generalise this framework and propose new numerical methods which leverage efficient classical algorithms for computing shortest paths on directed acyclic graphs. Our theoretical analysis confirms that these methods converge to globally optimal reconstructions. Numerically, we show new examples for unbalanced Optimal Transport penalties, and for balanced examples we are 100 times faster in comparison to the previously known method.

6.17 Inclusion and estimates for the jumps of minimizers in variational denoising

Participants: Antonin Chambolle, Michał Łasica.

We study in [12] the stability and inclusion of the jump set of minimizers of convex denoising functionals, such as the celebrated "Rudin-Osher-Fatemi" functional, for scalar or vectorial signals. We show that under mild regularity assumptions on the data fidelity term and the regularizer, the jump set of the minimizer is essentially a subset of the original jump set. Moreover, we give an estimate on the magnitude of jumps in terms of the data. This extends old results, in particular of the first author (with V. Caselles and M. Novaga) and of T. Valkonen, to much more general cases. We also consider the case where the original datum has unbounded variation, and define a notion of its jump set which, again, must contain the jump set of the solution.

6.18 L1-Gradient Flow of Convex Functionals

Participants: Antonin Chambolle, Matteo Novaga.

In [13], we are interested in the gradient flow of a general first order convex functional with respect to the L^1-topology. By means of an implicit minimization scheme, we show existence of a global limit solution, which satisfies an energy-dissipation estimate, and solves a non-linear and non-local gradient flow equation, under the assumption of strong convexity of the energy. Under a monotonicity assumption we can also prove uniqueness of the limit solution, even though this remains an open question in full generality. We also consider a geometric evolution corresponding to the L^1-gradient flow of the anisotropic perimeter. When the initial set is convex, we show that the limit solution is monotone for the inclusion, convex and unique until it reaches the Cheeger set of the initial datum. Eventually, we show with some examples that uniqueness cannot be expected in general in the geometric case.

6.19 Convergent plug-and-play with proximal denoiser and unconstrained regularization parameter

Participants: Antonin Chambolle, Samuel Hurault, Arthur Leclaire, Nicolas Papadakis.

In [17], we present new proofs of convergence for Plug-and-Play (PnP) algorithms. PnP methods are efficient iterative algorithms for solving image inverse problems where regularization is performed by plugging a pre-trained denoiser in a proximal algorithm, such as Proximal Gradient Descent (PGD) or Douglas-Rachford Splitting (DRS). Recent research has explored convergence by incorporating a denoiser that writes exactly as a proximal operator. However, the corresponding PnP algorithm has then to be run with stepsize equal to 1. The stepsize condition for nonconvex convergence of the proximal algorithm in use then translates to restrictive conditions on the regularization parameter of the inverse problem. This can severely degrade the restoration capacity of the algorithm. In this paper, we present two remedies for this limitation. First, we provide a novel convergence proof for PnP-DRS that does not impose any restrictions on the regularization parameter. Second, we examine a relaxed version of the PGD algorithm that converges across a broader range of regularization parameters. Our experimental study, conducted on deblurring and super-resolution experiments, demonstrate that both of these solutions enhance the accuracy of image restoration.

6.20 A Cahn–Hilliard–Willmore phase field model for non-oriented interfaces

Participants: Antonin Chambolle, Elie Bretin, Simon Masnou.

In [26], we investigate a new phase field model for representing non-oriented interfaces, approximating their area and simulating their area-minimizing flow. Our contribution is related to the approach proposed in arXiv:2105.09627 that involves ad hoc neural networks. We show here that, instead of neural networks, similar results can be obtained using a more standard variational approach that combines a Cahn-Hilliard-type functional involving an appropriate non-smooth potential and a Willmore-type stabilization energy. We show some properties of this phase field model in dimension 1 and, for radially symmetric functions, in arbitrary dimension. We propose a simple numerical scheme to approximate its L2-gradient flow. We illustrate numerically that the new flow approximates fairly well the mean curvature flow of codimension 1 or 2 interfaces in dimensions 2 and 3.

6.21 Phase-field approximation for 1-dimensional shape optimization problems

Participants: Joao-Miguel Machado.

In [32], we propose an unified framework for the phase field approximation of 1-dimensional shape optimization problems with connectedness constraints in any dimension. In particular, we focus on the average distance minimizers problem and the Wasserstein-H^1 problem recently introduced by Duval, Chambolle and Machado. The scheme relies on the p-Ambrosio-Tortorelli energy and the diffuse connectedness functional proposed by Dondl et al. that penalizes how disconnected the level sets of phase fields are. We argue that choosing p>d, not only the optimal profiles coming from the Ambrosio Tortorelli term present sharper transitions, but it also allows us to control the level sets of phase fields, enabling the analysis of the connectedness functional. This leads to general Γ-liminf and limsup inequalities that are easily adaptable to prove Γ-convergence results for the average distance and Wasserstein-H^1 problems.

7 Bilateral contracts and grants with industry

Participants: Jean-David Benamou, Gregoire Loeper.

CIFRE PhD thesis scholarship (Guillaume Chazareix) with BNP. Main supervisor Jean-David Benamou, co-supervision with Guillaume Carlier (Inria Mokaplan) and Gregoire Loeper (BNP). This contract is handled by Dauphine University.

8 Partnerships and cooperations

8.1 International research visitors

8.1.1 Visits of international scientists

Other international visits to the team

Participants: Guillaume Carlier, Jean-David Benamou, Paul Pegon.

Luigi De Pascale

Status Associate Professor

Institution of origin: Universita degli Studi Firenze

Country: Italie

Dates: 06/03–06/07, 06/26–07-10, 09/09–09/13 (the visit has been split because of the relocation of our offices)

Context of the visit: Luigi De Pascale is a long term collaborator of our team. He visited us to work on ongoing projects with Guillaume Carlier and Paul Pegon on L^∞-type optimal transport.

Mobility program/type of mobility: research stay

Jacob Francis

Status PhD student

Institution of origin: Imperial College London

Country: Royaume-Uni

Dates: From March, 4th to May, 10th

Context of the visit: This is part of an ongoing project on the use of Entropic Optimal Transport in the context of the semi-geostrophic approximation of rotating fluids. The research also involves Hugo Malamut, Colin Cotter (ICL) and Jean-David Benamou

Mobility program/type of mobility: research stay

8.1.2 Visits to international teams

Research stays abroad

Jean-David Benamou

Visited institution: Imperial College

Country: Royaume-Uni

Dates: Nov. 12th - Dec 11th

Context of the visit: This is part of an ongoing project on the use of Entropic Optimal Transport in the context of the semi-geostrophic approximation of rotating fluids. The research also involves Hugo Malamut Colin Cotter and Jacob Francis (ICL).

Mobility program/type of mobility: CNRS Imperial Abraham de Moivre Fellowship.

8.2 National initiatives

ANR CIPRESSI (2019-2024) is a JCJC grant (149k€) carried by Vincent Duval. Its aim is to develop off-the-grid methods for inverse problems involving the reconstruction of complex objects.

PDE AI (2023-2027) Antonin Chambolle is the main coordinator of the PDE-AI project, funded by the PEPR IA (France 2030, ANR) and gathering 10 groups throughout France working on PDEs and nonlinear analysis for artificial intelligence.

ANR GOTA is a JCJC grant (253k€) carried by Luca Nenna (PI), Paul Pegon and Maxime Laborde, dealing with some generalizations and applications of Optimal Transport theory with a particular focus on three main topics: multi-marginal optimal transport, urban planning and ulti-population models, and multi-marginal entropic optimal transport.

9 Dissemination

9.1 Promoting scientific activities

9.1.1 Scientific events: organisation

General chair, scientific chair Jean-David Benamou organizer of : Stochastic Control, Optimal Transport, Calibration applications in Finance, workshop Carry le Rouet.

Member of the organizing committees Antonin Chambolle is a co-organizer (member of the organizing and scientific committee) of "PGMO days 2024", EDF Saclay, 19–20 Nov. 2024

Vincent Duval was a member of the organizing committee of the Workhop Julia & Optimization Days 2024 in Toulouse in October 2024 ($\approx$ 60 participants). Vincent Duval is a member of the Imaging in Paris seminar (monthly seminar) organizing committee.

Guillaume Carlier coorganized with Eric Séré a two days conference in september in honor of Ivar Ekeland at Dauphine, a two days workshop with Alessio Figalli, Quentin Mérigot and Filippo Santambrogio on Particle systems and dynamics, optimization and learning at the Lagrange Center in march and is a coorganizer of the (monthly) séminaire parisien d'optimisation.

Paul Pegon and Vincent Duval coorganized the workshop for the 10th anniversary of Mokaplan.

9.1.2 Scientific events: selection

Reviewer

- Antonin Chambolle is a reviewer for AISTATS 2025 and SSVM 2025

9.1.3 Journal

Member of the editorial boards Antonin Chambolle is in the editorial board of

- Interfaces and Free Boundaries (EMS Press, one of the 4 co-editors in chief)
- Journal of the European Math. Society (JEMS, EMS Press)
- M2AN Mathematical Modeling and Numerical Analysis
- AMO Applied Math. Optim (Springer)
- IPI Inverse problems and imaging (AIMS)
- JMIV Journal of Math. Imaging and Vision (Springer)
- A special issue of JOTA (Journal of Optimization Theory and Applications, Springer, special issue on optimization for energy).

Vincent Duval is associate editor for the Journal of Mathematical Imaging and Vision journal (JMIV).

Guillaume Carlier is associate editor of Journal of Mathematical Analysis and Applications, Applied Mathematics and Optimization, Journal of Dynamics and Games.

Reviewer - reviewing activities

- Irène Waldspurger: Mathematical Programming.
- Vincent Duval: Computational Optimization and Applications, SIAM Journal on Imaging Sciences, Mathematical reviews (x2).

9.1.4 Invited talks

- Flavien Léger: Journée SIGMA – MODE, INRIA Paris, January 2024 ;
- Flavien Léger: Learning and Optimization in Luminy, CIRM, June 2024 ;
- Flavien Léger: Math+Econ+Code seminar, Paris, October 2024 ;
- Flavien Léger: Séminaire du MIA Paris-Saclay, November 2024 ;
- Flavien Léger: Optimal Transportation and Applications, Pisa, December 2024 ;
- Flavien Léger: Online seminar on Geometry and Statistics, December 2024 ;
- Irène Waldspurger : journée thématique *Low-rank approximation and optimization*, at the Institut de mathématiques de Marseille ;
- Irène Waldspurger : session on *Negative eigenvalues and nonconvex optimization*, at the SIAM conference on Applied Linear Algebra (Paris) ;
- Irène Waldspurger : *Women in Machine Learning and Data Science* event ;
- Guillaume Carlier: Colloquium Créteil, Workshop on Optimal Transport from Theory to Applications, Berlin, TSE Madstat Seminar, Toulouse, GT Calva à Orsay, Seminar on Optimization, University of Vienna, Nonlinear Analysis seminar, ISTA, Vienna, Optimal Transportation and Applications Centro di Ricerca Matematica Ennio De Giorgi, Pisa.
- Paul Pegon: Analysis Seminar, École Polytechnique (Palaiseau), March 2024 ;
- Paul Pegon: GT *Optimal Transport and Optimisation*, Université Aix-Marseille and Université de Toulon, April 2024 ;
- Paul Pegon: Mini-symposium at CANUM 2024, Université de La Rochelle, May 2024 ;
- Jean-David Benamou : Numerical methods for optimal transport problems, mean field games and multi-agent dynamics, Universidad Federico Santa María in Valparaíso, Chile.
- Jean-David Benamou : MFO Workshop 2406 on Optimal Transportation (Oberwolfach).
- Jean-David Benamou : MOKA10 workshop (ten years of Mokaplan), Fontainebleau .
- Jean-David Benamou : Optimal transport workshop, IES CNRS, Cargese.
- Vincent Duval : Journée SIGMA – MODE, INRIA Paris, January 2024 ;
- Antonin Chambolle was invited, and gave a seminar a the MFO Oberwolfach workshop #2407, “Interfaces, Free Boundaries and Geometric Partial Differential Equations”, 11–16 Feb. 2024 (talk on “Discrete-to-continuum crystalline curvature flow”, 12 Feb. 2024).
- Antonin Chambolle was invited to give a talk at the (French) mini-workshop “a 3×60 tour of optimal design”, Autrans, 6–8 March 2024, in honor of Grégoire Allaire, Eric Bonnetier, François Jouve.
- Antonin Chambolle gave a plenary talk at the conference “ICIPE 24” (“Internat. Conf. on Inverse Problems in Engineering”), Buzios, Brazil, 23–27 June 2024, on “Qualitative Properties of Minimizers of Total-Variation Regularized Problems” (26 June).

- Antonin Chambolle gave an invited talk at the "IFIP TC7 Meeting on System Modeling and Optimization" in Hamburg, 12–16 Aug. 2024, "Qualitative Properties of Minimizers of Total-Variation Regularized Problems" (13 Aug.).
- Antonin Chambolle gave an invited talk at "VARANA 2024", Erice, Italie, on "Discrete-to-continuum crystalline curvature flow", 6 Sept.
- Antonin Chambolle participated to the "Conférence pour les 50 ans du CMAP", 11–13 Sept. 2024 and gave a talk on "Discrete-to-continuum crystalline curvature flow" (13 Sept.).
- Antonin Chambolle gave a talk at the workshop "Mathematical Materials Science: Defects and Polycrystals", Edinburgh, 13–19 Oct. 2024, on "Discrete-to-continuum crystalline curvature flow" (15 Oct.).

9.1.5 Research administration

- Antonin Chambolle is a member of the scientific council of Université Paris-Dauphine. He is also a member of the scientfic council and of the board of the PGMO "Programme Gaspard Monge pour l'Optimisation et la Recherche Opérationnelle".
- Irène Waldspurger : hiring committee for a maîtresse de conférence position at INSA Rennes.
- Vincent Duval is "délégué scientifique adjoint" of the Inria Paris center.
- Vincent Duval was vice-president of the CRCN-ISFP selection committee of the Inria Paris center, and a member of the selection committee for the DR2 Inria selection.
- Guillaume Carlier is in the scientific board of the PGMO
- Paul Pegon was a member of a hiring committee for a Maître de conférences position at Université de Toulon

9.2 Teaching - Supervision - Juries

9.2.1 Teaching

- Master: Antonin Chambolle Optimisation Continue, 24h, niveau M2, Université Paris Dauphine-PSL, FR
- Master : Vincent Duval, Problèmes Inverses, 22,5 h équivalent TD, niveau M1, Université PSL/Mines ParisTech, FR
- Licence : Irène Waldspurger, Algorithmique et programmation, 34,5 h équivalent TD, niveau L2, Université Paris-Dauphine, FR
- Master : Irène Waldspurger, Optimization for Machine Learning, 6h, niveau M2, Université PSL/ENS, FR
- Master : Irène Waldspurger, Introduction à la géométrie différentielle et aux équations différentielles, 29,25 h équivalent TD, niveau M1, Université Paris Dauphine, FR
- Master : Irène Waldspurger, Non-convex inverse problems, 27 h d'équivalent TD, niveau M2, Université Paris Dauphine, FR
- Licence : Guillaume Carlier, algebre 1, L1 78h, Dauphine, FR
- Master/PhD : Guillaume Carlier Optimal transport, University of Vienna, 21h
- Guillaume Carlier: Licence Algèbre 1, Dauphine 70h, M2 Masef: Variational and transport problems in economics, 18h
- Licence : Paul Pegon, Analysis 1, L1, 146h eq TD, FR
- Licence : Paul Pegon, Analysis 4, L1, 50h eq TD, FR

9.2.2 Supervision

- L3 memoir completed: Enzo Regna (Univ Paris-Dauphine) supervised by Paul Pegon.
- M2 internship completed: Majid Arthaud (ENPC), co-supervised by Antonin Chambolle and Vincent Duval.
- M2 internship completed: Paul Caucheteux (master MASH, Dauphine) supervised by Irène Waldspurger.
- M2 internship completed: Lorenzo Braglia (Università di Padova), co-supervised by Paul Pegon and Antonin Chambolle.
- M2 internship completed: Louis Tocquec (ENS Paris-Saclay), co-supervised by Paul Pegon and Luca Nenna.
- Phd completed: Joao Pinto Anastacio Machado, *Transport optimal et structures géométriques*, defended on 17/09/2024, co-supervised by Antonin Chambolle and Vincent Duval.
- PhD in progress: Faniriana Rakoto Endor, *Pourquoi l'heuristique de Burer-Monteiro fonctionne-t-elle si bien?*, started on 01/10/2023, co-supervised by Irène Waldspurger and Antonin Chambolle.
- Phd in progress: Hugo Malamut, *Régularisation Entropique et Transport Optimal Généralisé*, started on 1/09/2022, co-supervised by Jean-David Benamou and Guillaume Carlier.
- Phd in progress: Maxime Sylvestre, *On Hybrid methods fot Optimal Transport*, started on 01/09/2022, co-supervised by Guillaume Carlier and Alfred Galichon.
- Phd in progress: Guillaume Chazareix, *Non Linear Parabolic equations and Volatility Calibration*, started on 1/08/2021, co-supervised by Jean-David Benamou and Grégoire Loeper.
- PhD in progress: Louis Tocquec, *On entropic optimal transport and some generalizations*, started on 01/10/2024, co-supervised by Paul Pegon and Luca Nenna.
- Postdoc completed: Adrien Vacher, supervised by Flavien Léger.

9.2.3 Juries

- Vincent Duval was a jury member for the PhD of Théo Bertrand (Université Paris-Dauphine PSL)
- Irène Waldspurger was a jury member for the PhD of Jean-Jacques Godème (ENSICAEN).
- Irène Waldspurger was a jury member for the PhD of Pierre-Jean Bénard (Institut Mathématique de Bordeaux).
- Guillaume Carlier was the coordinator of the HDR of Thomas Gallouët and a referee for the PhD's of Jules Candau-Tilh (Lille) and Giacomo Greco (Eindhoven, Netherlands).

9.3 Popularization

9.3.1 Productions (articles, videos, podcasts, serious games, ...)

Vincent Duval has worked together with a cartoonist (Julien Joliclerc) for the writing of a popularization cartoon on his ANR JCJC project, for the ANR Medianum project.

9.3.2 Participation in Live events

Flavien Léger gave the "demi-heure de science" at inria Paris, October 2024.

10 Scientific production

10.1 Major publications

[1] P.-C. Aubin-Frankowski, A. Korba and F. Léger. 'Mirror Descent with Relative Smoothness in Measure Spaces, with application to Sinkhorn and EM'. In: NeurIPS 2022 - Thirty-sixth Conference on Neural Information Processing Systems. New Orleans, United States, 2022. URL: https://hal.science/hal-03811583.

[2] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna and G. Peyré. 'Iterative Bregman Projections for Regularized Transportation Problems'. In: *SIAM Journal on Scientific Computing* 2.37 (2015), A1111–A1138. DOI: 10.1137/141000439. URL: https://hal.science/hal-01096124.

[3] J.-D. Benamou, T. Gallouët and F.-X. Vialard. 'Second order models for optimal transport and cubic splines on the Wasserstein space'. In: *Foundations of Computational Mathematics* (Oct. 2019). DOI: 10.1007/s10208-019-09425-z. URL: https://hal.science/hal-01682107.

[4] C. Boyer, A. Chambolle, Y. de Castro, V. Duval, F. de Gournay and P. Weiss. 'On Representer Theorems and Convex Regularization'. In: *SIAM Journal on Optimization* 29.2 (9th May 2019), pp. 1260–1281. DOI: 10.1137/18M1200750. URL: https://hal.archives-ouvertes.fr/hal-01823135.

[5] C. Cancès, T. Gallouët and G. Todeschi. 'A variational finite volume scheme for Wasserstein gradient flows'. In: *Numerische Mathematik* 146.3 (2020), pp 437 –480. DOI: 10.1007/s00211-020-01153-9. URL: https://hal.science/hal-02189050.

[6] G. Carlier, V. Duval, G. Peyré and B. Schmitzer. 'Convergence of Entropic Schemes for Optimal Transport and Gradient Flows'. In: *SIAM Journal on Mathematical Analysis* 49.2 (18th Apr. 2017). DOI: 10.1137/15M1050264. URL: https://hal.science/hal-01246086.

[7] G. Carlier, P. Pegon and L. Tamanini. *Convergence rate of general entropic optimal transport costs*. 7th June 2022. URL: https://hal.archives-ouvertes.fr/hal-03689945.

[8] F. Léger and P.-C. Aubin-Frankowski. *Gradient descent with a general cost*. 14th Dec. 2023. URL: https://hal.science/hal-04344054.

[9] F. Léger and F.-X. Vialard. *A geometric Laplace method*. 22nd Dec. 2022. URL: https://hal.science/hal-03911149 (cit. on p. 11).

[10] I. Waldspurger. 'Phase retrieval with random Gaussian sensing vectors by alternating projections'. In: *IEEE Transactions on Information Theory* 64.5 (2018), pp. 3301–3312. URL: https://hal.science/hal-01645081.

[11] I. Waldspurger and A. Waters. 'Rank optimality for the Burer-Monteiro factorization'. In: *SIAM Journal on Optimization* 30.3 (2020), pp. 2577–2602. DOI: 10.1137/19M1255318. URL: https://hal.science/hal-01958814.

10.2 Publications of the year

International journals

[12] A. Chambolle and M. Łasica. 'Inclusion and estimates for the jumps of minimizers in variational denoising'. In: *SIAM Journal on Imaging Sciences* 17.3 (1st Sept. 2024), pp. 1844–1878. DOI: 10.1137/23M1627948. URL: https://hal.science/hal-04323807 (cit. on p. 15).

[13] A. Chambolle and M. Novaga. 'L1-Gradient Flow of Convex Functionals'. In: *SIAM Journal on Mathematical Analysis* 56.5 (3rd Sept. 2024), pp. 5747–5781. DOI: 10.1137/22M1527556. URL: https://hal.science/hal-03805962 (cit. on p. 15).

[14] V. Duval and R. Tovey. 'Dynamical programming for off-the-grid dynamic inverse problems'. In: *ESAIM: Control, Optimisation and Calculus of Variations* 30 (2024), p. 7. DOI: 10.1051/cocv/2023085. URL: https://hal.science/hal-04450197 (cit. on p. 15).

[15] E. Facca, G. Todeschi, A. Natale and M. Benzi. 'Efficient preconditioners for solving dynamical optimal transport via interior point methods'. In: *SIAM Journal on Scientific Computing* 46.3 (2024). DOI: 10.1137/23M1570430. URL: https://inria.hal.science/hal-03766668.

[16] T. Gallouët, A. Natale and G. Todeschi. 'From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows'. In: *Mathematics of Computation* 93 (2024), pp. 2769–2810. DOI: 10.1090/mcom/3951. URL: https://hal.science/hal-03790981.

[17] S. Hurault, A. Chambolle, A. Leclaire and N. Papadakis. 'Convergent plug-and-play with proximal denoiser and unconstrained regularization parameter'. In: *Journal of Mathematical Imaging and Vision* (2024). URL: https://hal.science/hal-04269033. In press (cit. on p. 16).

[18] A. Monteil and P. Pegon. 'Mass concentration in rescaled first order integral functionals'. In: *Journal de l'École polytechnique — Mathématiques* 11 (19th Feb. 2024), p. 42. DOI: 10.5802/jep.257. URL: https://hal.science/hal-03517074 (cit. on p. 12).

[19] B. Muzellec, A. Vacher, F. Bach, F.-X. Vialard and A. Rudi. 'Near-optimal estimation of smooth transport maps with kernel sums-of-squares'. In: *SIAM Journal on Mathematics of Data Science* (2024). URL: https://hal.science/hal-03466696. In press.

[20] L. Nenna and P. Pegon. 'Convergence rate of entropy-regularized multi-marginal optimal transport costs'. In: *Canadian Journal of Mathematics = Journal Canadien de Mathématiques* (15th Mar. 2024). DOI: 10.4153/S0008414X24000257. URL: https://hal.science/hal-04154453 (cit. on p. 11).

[21] A. Sportisse, M. Marbac, F. Laporte, G. Celeux, C. Boyer, J. Josse and C. Biernacki. 'Model-based Clustering with Missing Not At Random Data'. In: *Statistics and Computing* (18th June 2024). DOI: 10.1007/s11222-024-10444-2. URL: https://hal.science/hal-03494674.

[22] M. Zach, E. Kobler, A. Chambolle and T. Pock. 'Product of Gaussian Mixture Diffusion Models'. In: *Journal of Mathematical Imaging and Vision* 66.4 (15th Mar. 2024), pp. 504–528. DOI: 10.1007/s10851-024-01180-3. URL: https://hal.science/hal-04790492.

Reports & preprints

[23] J.-D. Benamou, G. Chazareix, M. Hoffmann, G. Loeper and F.-X. Vialard. *Entropic Semi-Martingale Optimal Transport*. 2024. DOI: 10.48550/arXiv.2408.09361. URL: https://hal.science/hal-04673273 (cit. on p. 11).

[24] J.-D. Benamou, G. Chazareix and G. Loeper. *From entropic transport to martingale transport, and applications to model calibration*. 17th June 2024. URL: https://hal.science/hal-04613721 (cit. on p. 11).

[25] S. Boufadène and F.-X. Vialard. *On the global convergence of Wasserstein gradient flow of the Coulomb discrepancy*. 26th Jan. 2024. URL: https://hal.science/hal-04282762.

[26] É. Bretin, A. Chambolle and S. Masnou. *A Cahn–Hilliard–Willmore phase field model for non-oriented interfaces*. 2024. URL: https://hal.science/hal-04884780 (cit. on p. 16).

[27] G. Carlier, L. Chizat and M. Laborde. *Displacement smoothness of entropic optimal transport*. 1st Mar. 2024. URL: https://hal.science/hal-03793562.

[28] G. Carlier and H. Malamut. *Well-posedness and convergence of entropic approximation of semi-geostrophic equations*. 25th Apr. 2024. URL: https://hal.science/hal-04560854.

[29] A. Chambolle, D. de Gennaro and M. Morini. *Discrete-to-continuous crystalline curvature flows*. 7th Mar. 2024. URL: https://hal.science/hal-04500307.

[30] A. Chambolle and J.-M. Morel. *Image = Cartoon+Texture: How Yves Meyer's "Oscillating patterns in image processing and in some nonlinear evolution equations" ended up in a computer vision model*. 2024. URL: https://hal.science/hal-04893423.

[31] F. R. Endor and I. Waldspurger. *Benign landscape for Burer-Monteiro factorizations of MaxCut-type semidefinite programs*. 5th Nov. 2024. URL: https://hal.science/hal-04797879 (cit. on p. 14).

[32] J. M. Machado. *Phase-field approximation for 1-dimensional shape optimization problems*. 21st June 2024. URL: https://hal.science/hal-04620380 (cit. on p. 16).

[33] G. Olikier and I. Waldspurger. *Projected gradient descent accumulates at Bouligand stationary points*. 4th Mar. 2024. URL: https://hal.science/hal-04588622 (cit. on p. 14).

10.3 Cited publications

[34] M. Agueh and G. Carlier. 'Barycenters in the Wasserstein space'. In: *SIAM J. Math. Anal.* 43.2 (2011), pp. 904–924 (cit. on p. 4).

[35] L. Ambrosio and W. Gangbo. 'Hamiltonian ODEs in the Wasserstein space of probability measures'. In: *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 61.1 (2008), pp. 18–53 (cit. on p. 8).

[36] D. N. Arnold, R. S. Falk and R. Winther. 'Finite element exterior calculus, homological techniques, and applications'. In: *Acta Numerica* 15 (2006), pp. 1–155. DOI: 10.1017/S0962492906210018 (cit. on p. 10).

[37] F. R. Bach. 'Consistency of the Group Lasso and Multiple Kernel Learning'. In: *J. Mach. Learn. Res.* 9 (June 2008), pp. 1179–1225. URL: http://dl.acm.org/citation.cfm?id=1390681.1390721 (cit. on p. 6).

[38] F. R. Bach. 'Consistency of Trace Norm Minimization'. In: *J. Mach. Learn. Res.* 9 (June 2008), pp. 1019–1048. URL: http://dl.acm.org/citation.cfm?id=1390681.1390716 (cit. on p. 6).

[39] J. D. Backhoff-Veraguas and G. Pammer. *Applications of weak transport theory*. 2020. arXiv: 2003.05338 [math.PR] (cit. on p. 8).

[40] X. Bacon, G. G. Carlier and B. Nazaret. 'A spatial Pareto exchange economy problem'. working paper or preprint. Dec. 2021. URL: https://hal.science/hal-03480323 (cit. on p. 9).

[41] C. Barilla, G. Carlier and J.-M. Lasry. 'A mean field game model for the evolution of cities'. In: *Journal of Dynamics and Games* (2021). URL: https://hal.science/hal-03086616 (cit. on p. 9).

[42] M. F. Beg, M. I. Miller, A. Trouvé and L. Younes. 'Computing Large Deformation Metric Mappings via Geodesic Flows of Diffeomorphisms'. In: *International Journal of Computer Vision* 61.2 (Feb. 2005), pp. 139–157. URL: http://dx.doi.org/10.1023/B:VISI.0000043755.93987.aa (cit. on p. 5).

[43] M. Beiglbock, P. Henry-Labordère and F. Penkner. 'Model-independent bounds for option prices mass transport approach'. English. In: *Finance and Stochastics* 17.3 (2013), pp. 477–501. DOI: 10.1007/s00780-013-0205-8. URL: http://dx.doi.org/10.1007/s00780-013-0205-8 (cit. on p. 4).

[44] J.-D. Benamou and Y. Brenier. 'A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem'. In: *Numer. Math.* 84.3 (2000), pp. 375–393. DOI: 10.1007/s002110050002. URL: http://dx.doi.org/10.1007/s002110050002 (cit. on pp. 4, 7).

[45] J.-D. Benamou and Y. Brenier. 'Weak existence for the semigeostrophic equations formulated as a coupled Monge-Ampère/transport problem'. In: *SIAM J. Appl. Math.* 58.5 (1998), pp. 1450–1461 (cit. on pp. 4, 8).

[46] J.-D. Benamou and G. Carlier. 'Augmented Lagrangian algorithms for variational problems with divergence constraints'. In: *JOTA* (2015) (cit. on p. 4).

[47] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna and G. Peyré. 'Iterative Bregman Projections for Regularized Transportation Problems'. In: *SIAM J. Sci. Comp.* (2015). to appear (cit. on pp. 4, 7).

[48] J.-D. Benamou, G. Chazareix, G. Rukhaia and W. L. Ijzerman. 'Point Source Regularization of the Finite Source Reflector Problem'. In: *Journal of Computational Physics* (May 2022). URL: https://inria.hal.science/hal-03344571 (cit. on p. 9).

[49] J.-D. Benamou, B. D. Froese and A. Oberman. 'Two numerical methods for the elliptic Monge-Ampère equation'. In: *M2AN Math. Model. Numer. Anal.* 44.4 (2010), pp. 737–758 (cit. on p. 3).

[50] J.-D. Benamou, B. D. Froese and A. Oberman. 'Numerical solution of the optimal transportation problem using the Monge–Ampere equation'. In: *Journal of Computational Physics* 260 (2014), pp. 107–126 (cit. on p. 3).

[51] J. Bigot and T. Klein. 'Consistent estimation of a population barycenter in the Wasserstein space'. In: *Preprint arXiv:1212.2562* (2012) (cit. on p. 4).

[52] M. Boissier, G. Allaire and C. Tournier. 'Additive manufacturing scanning paths optimization using shape optimization tools'. In: *Struct. Multidiscip. Optim.* 61.6 (2020), pp. 2437–2466. DOI: 10.1007/s00158-020-02614-3. URL: https://doi.org/10.1007/s00158-020-02614-3 (cit. on p. 9).

[53] N. Bonneel, J. Rabin, G. Peyré and H. Pfister. 'Sliced and Radon Wasserstein Barycenters of Measures'. In: *Journal of Mathematical Imaging and Vision* 51.1 (2015), pp. 22–45. URL: http://hal.archives-ouvertes.fr/hal-00881872/ (cit. on p. 4).

[54] G. Bouchitté and G. Buttazzo. 'Characterization of optimal shapes and masses through Monge-Kantorovich equation'. In: *J. Eur. Math. Soc. (JEMS)* 3.2 (2001), pp. 139–168. DOI: 10.1007/s100970000027. URL: http://dx.doi.org/10.1007/s100970000027 (cit. on p. 5).

[55] N. Boumal, V. Voroninski and A. S. Bandeira. 'Deterministic guarantees for Burer-Monteiro factorizations of smooth semidefinite programs'. In: *preprint* (2018). https://arxiv.org/abs/1804.02008 (cit. on p. 9).

[56] K. Bredies, M. Carioni, S. Fanzon and F. Romero. 'A generalized conditional gradient method for dynamic inverse problems with optimal transport regularization'. In: *arXiv preprint arXiv:2012.11706* (2020) (cit. on p. 9).

[57] Y. Brenier. 'Décomposition polaire et réarrangement monotone des champs de vecteurs'. In: *C. R. Acad. Sci. Paris Sér. I Math.* 305.19 (1987), pp. 805–808 (cit. on p. 3).

[58] Y. Brenier. 'Generalized solutions and hydrostatic approximation of the Euler equations'. In: *Phys. D* 237.14-17 (2008), pp. 1982–1988. DOI: 10.1016/j.physd.2008.02.026. URL: http://dx.doi.org/10.1016/j.physd.2008.02.026 (cit. on p. 4).

[59] Y. Brenier. 'Polar factorization and monotone rearrangement of vector-valued functions'. In: *Comm. Pure Appl. Math.* 44.4 (1991), pp. 375–417. DOI: 10.1002/cpa.3160440402. URL: http://dx.doi.org/10.1002/cpa.3160440402 (cit. on p. 3).

[60] Y. Brenier, U. Frisch, M. Henon, G. Loeper, S. Matarrese, R. Mohayaee and A. Sobolevski. 'Reconstruction of the early universe as a convex optimization problem'. In: *Mon. Not. Roy. Astron. Soc.* 346 (2003), pp. 501–524. URL: http://arxiv.org/pdf/astro-ph/0304214.pdf (cit. on p. 4).

[61] M. Burger and S. Osher. 'A guide to the TV zoo'. In: *Level-Set and PDE-based Reconstruction Methods, Springer* (2013) (cit. on p. 6).

[62] G. Buttazzo, A. Pratelli, E. Stepanov and S. Solimini. *Optimal Urban Networks via Mass Transportation.* Vol. 1961. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009. DOI: 10.1007/978-3-540-85799-0. URL: http://link.springer.com/10.1007/978-3-540-85799-0 (visited on 11/01/2022) (cit. on p. 10).

[63] L. A. Caffarelli. 'The regularity of mappings with a convex potential'. In: *J. Amer. Math. Soc.* 5.1 (1992), pp. 99–104. DOI: 10.2307/2152752. URL: http://dx.doi.org/10.2307/2152752 (cit. on p. 3).

[64] L. A. Caffarelli, S. A. Kochengin and V. Oliker. 'On the numerical solution of the problem of reflector design with given far-field scattering data'. In: *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997).* Vol. 226. Contemp. Math. Providence, RI: Amer. Math. Soc., 1999, pp. 13–32. DOI: 10.1090/conm/226/03233. URL: http://dx.doi.org/10.1090/conm/226/03233 (cit. on p. 4).

[65] C. CanCeritoglu. 'Computational Analysis of LDDMM for Brain Mapping'. In: *Frontiers in Neuroscience* 7 (2013) (cit. on p. 5).

[66] E. Candes and M. Wakin. 'An Introduction to Compressive Sensing'. In: *IEEE Signal Processing Magazine* 25.2 (2008), pp. 21–30 (cit. on p. 6).

[67] E. J. Candès and C. Fernandez-Granda. 'Super-Resolution from Noisy Data'. In: *Journal of Fourier Analysis and Applications* 19.6 (2013), pp. 1229–1254 (cit. on p. 6).

[68] E. J. Candès and C. Fernandez-Granda. 'Towards a Mathematical Theory of Super-Resolution'. In: *Communications on Pure and Applied Mathematics* 67.6 (2014), pp. 906–956 (cit. on p. 6).

[69] G. Carlier, A. Dupuy, A. Galichon and Y. Sun. 'SISTA: Learning Optimal Transport Costs under Sparsity Constraints'. working paper or preprint. Oct. 2020. URL: https://hal.science/hal-03504045 (cit. on p. 8).

[70] G. Carlier, P. Pegon and L. Tamanini. 'Convergence rate of general entropic optimal transport costs'. In: *Calculus of Variations and Partial Differential Equations* 62.4 (May 2023), p. 116. DOI: 10.1007/s00526-023-02455-0. URL: https://hal.science/hal-03689945 (cit. on p. 11).

[71] G. Carlier and C. Poon. 'On the total variation Wasserstein gradient flow and the TV-JKO scheme'. In: *ESAIM: Control, Optimisation and Calculus of Variations* (2019). URL: https://hal.science/hal-01492343 (cit. on p. 8).

[72] Y. de Castro, V. Duval and R. Petit. 'Towards Off-the-grid Algorithms for Total Variation Regularized Inverse Problems'. In: *Journal of Mathematical Imaging and Vision* (July 2022). DOI: 10.1007/s10851-022-01115-w. URL: https://inria.hal.science/hal-03406710 (cit. on p. 9).

[73] F. A. C. C. Chalub, P. A. Markowich, B. Perthame and C. Schmeiser. 'Kinetic models for chemotaxis and their drift-diffusion limits'. In: *Monatsh. Math.* 142.1-2 (2004), pp. 123–141. DOI: 10.1007/s00605-004-0234-7. URL: http://dx.doi.org/10.1007/s00605-004-0234-7 (cit. on p. 5).

[74] A. Chambolle, L. A. D. Ferrari and B. Merlet. 'Variational approximation of size-mass energies for k-dimensional currents'. In: *ESAIM Control Optim. Calc. Var.* 25 (2019), Paper No. 43, 39. DOI: 10.1051/cocv/2018027. URL: https://doi.org/10.1051/cocv/2018027 (cit. on p. 10).

[75] A. Chambolle and T. Pock. 'Crouzeix-Raviart approximation of the total variation on simplicial meshes'. In: *J. Math. Imaging Vision* 62.6-7 (2020), pp. 872–899. DOI: 10.1007/s10851-019-00939-3. URL: https://doi.org/10.1007/s10851-019-00939-3 (cit. on p. 10).

[76] A. Chambolle and T. Pock. 'Learning consistent discretizations of the total variation'. In: *SIAM J. Imaging Sci.* 14.2 (2021), pp. 778–813. DOI: 10.1137/20M1377199. URL: https://doi.org/10.1137/20M1377199 (cit. on p. 10).

[77] T. Champion, L. De Pascale and P. Juutinen. 'The ∞-Wasserstein Distance: Local Solutions and Existence of Optimal Transport Maps'. In: *SIAM Journal on Mathematical Analysis* 40.1 (1st Jan. 2008), pp. 1–20. DOI: 10.1137/07069938X. URL: https://epubs.siam.org/doi/10.1137/07069938X (visited on 12/01/2022) (cit. on p. 8).

[78] S. S. Chen, D. L. Donoho and M. A. Saunders. 'Atomic decomposition by basis pursuit'. In: *SIAM journal on scientific computing* 20.1 (1999), pp. 33–61 (cit. on p. 6).

[79] L. Chizat, P. Roussillon, F. Léger, F.-X. Vialard and G. Peyré. 'Faster Wasserstein Distance Estimation with the Sinkhorn Divergence'. In: *Neural Information Processing Systems*. Advances in Neural Information Processing Systems. Vancouver, Canada, Dec. 2020. URL: https://hal.archives-ouvertes.fr/hal-02867271 (cit. on p. 7).

[80] L. Condat. 'Discrete total variation: new definition and minimization'. In: *SIAM J. Imaging Sci.* 10.3 (2017), pp. 1258–1290. URL: https://doi.org/10.1137/16M1075247 (cit. on p. 10).

[81] C. Cotar, G. Friesecke and C. Kluppelberg. 'Density Functional Theory and Optimal Transportation with Coulomb Cost'. In: *Communications on Pure and Applied Mathematics* 66.4 (2013), pp. 548–599. DOI: 10.1002/cpa.21437. URL: http://dx.doi.org/10.1002/cpa.21437 (cit. on p. 3).

[82] M. J. P. Cullen. *A Mathematical Theory of Large-Scale Atmosphere/Ocean Flow*. Imperial College Press, 2006. URL: https://books.google.fr/books?id=JxBqDQAAQBAJ (cit. on p. 8).

[83] M. J. P. Cullen, W. Gangbo and G. Pisante. 'The semigeostrophic equations discretized in reference and dual variables'. In: *Arch. Ration. Mech. Anal.* 185.2 (2007), pp. 341–363. DOI: 10.1007/s00205-006-0040-6. URL: http://dx.doi.org/10.1007/s00205-006-0040-6 (cit. on p. 4).

[84] M. J. P. Cullen, J. Norbury and R. J. Purser. 'Generalised Lagrangian solutions for atmospheric and oceanic flows'. In: *SIAM J. Appl. Math.* 51.1 (1991), pp. 20–31 (cit. on p. 4).

[85] M. Cuturi. 'Sinkhorn Distances: Lightspeed Computation of Optimal Transport'. In: *Proc. NIPS*. Ed. by C. J. C. Burges, L. Bottou, Z. Ghahramani and K. Q. Weinberger. 2013, pp. 2292–2300 (cit. on pp. 4, 7).

[86] L. De Pascale and J. Louet. 'A Study of the Dual Problem of the One-Dimensional L^∞-Optimal Transport Problem with Applications'. In: *Journal of Functional Analysis* 276.11 (1st June 2019), pp. 3304–3324. DOI: 10.1016/j.jfa.2019.02.014. URL: https://www.sciencedirect.com/science/article/pii/S0022123619300643 (visited on 12/01/2022) (cit. on p. 8).

[87] E. J. Dean and R. Glowinski. 'Numerical methods for fully nonlinear elliptic equations of the Monge-Ampère type'. In: *Comput. Methods Appl. Mech. Engrg.* 195.13-16 (2006), pp. 1344–1386 (cit. on p. 3).

[88] V. Duval and G. Peyré. 'Exact Support Recovery for Sparse Spikes Deconvolution'. English. In: *Foundations of Computational Mathematics* (2014), pp. 1–41. DOI: 10.1007/s10208-014-9228-6. URL: http://dx.doi.org/10.1007/s10208-014-9228-6 (cit. on p. 6).

[89] C. Fernandez-Granda. 'Support detection in super-resolution'. In: *Proc. Proceedings of the 10th International Conference on Sampling Theory and Applications* (2013), pp. 145–148 (cit. on p. 6).

[90] J. Feydy, T. Séjourné, F.-X. Vialard, S.-I. Amari, A. Trouvé and G. Peyré. 'Interpolating between Optimal Transport and MMD using Sinkhorn Divergences'. working paper or preprint. Oct. 2018. URL: https://hal.science/hal-01898858 (cit. on p. 7).

[91] G. Friesecke and D. Vögler. 'Breaking the Curse of Dimension in Multi-Marginal Kantorovich Optimal Transport on Finite State Spaces'. In: *SIAM Journal on Mathematical Analysis* 50.4 (2018), pp. 3996–4019. DOI: 10.1137/17M1150025. eprint: https://doi.org/10.1137/17M1150025. URL: https://doi.org/10.1137/17M1150025 (cit. on p. 8).

[92] U. Frisch, S. Matarrese, R. Mohayaee and A. Sobolevski. 'Monge-Ampère-Kantorovitch (MAK) reconstruction of the eary universe'. In: *Nature* 417.260 (2002) (cit. on p. 4).

[93] A. Galichon, P. Henry-Labordère and N. Touzi. 'A stochastic control approach to No-Arbitrage bounds given marginals, with an application to Loopback options'. In: *submitted to Annals of Applied Probability* (2011) (cit. on p. 4).

[94] W. Gangbo and R. McCann. 'The geometry of optimal transportation'. In: *Acta Math.* 177.2 (1996), pp. 113–161. DOI: 10.1007/BF02392620. URL: http://dx.doi.org/10.1007/BF02392620 (cit. on p. 3).

[95] E. Ghys. 'Gaspard Monge, Le mémoire sur les déblais et les remblais'. In: *Image des mathématiques, CNRS* (2012). URL: http://images.math.cnrs.fr/Gaspard-Monge,1094.html (cit. on p. 3).

[96] I. Guo and G. Loeper. 'Path Dependent Optimal Transport and Model Calibration on Exotic Derivatives'. In: *SSRN Electron. J.* (Jan. 2018). Available at doi:10.2139/ssrn.3302384. DOI: 10.2139/ssrn.3302384 (cit. on p. 8).

[97] J. Haskovec, P. Markowich, B. Perthame and M. Schlottbom. 'Notes on a PDE System for Biological Network Formation'. In: *Nonlinear Analysis*. Nonlinear Partial Differential Equations, in Honor of Juan Luis Vázquez for His 70th Birthday 138 (1st June 2016), pp. 127–155. DOI: 10.1016/j.na.2015.12.018. URL: https://www.sciencedirect.com/science/article/pii/S0362546X15004344 (visited on 10/06/2021) (cit. on p. 10).

[98] D. D. Holm, J. T. Ratnanather, A. Trouvé and L. Younes. 'Soliton dynamics in computational anatomy'. In: *NeuroImage* 23 (2004), S170–S178 (cit. on p. 5).

[99] B. J. Hoskins. 'The mathematical theory of frontogenesis'. In: *Annual review of fluid mechanics, Vol. 14*. Palo Alto, CA: Annual Reviews, 1982, pp. 131–151 (cit. on p. 4).

[100] M. Jacobs, W. Lee and F. Léger. 'The back-and-forth method for Wasserstein gradient flows'. In: *ESAIM: Control, Optimisation and Calculus of Variations* 27 (2021), p. 28 (cit. on p. 8).

[101] M. Jacobs and F. Léger. 'A fast approach to optimal transport: The back-and-forth method'. In: *Numer. Math.* 146 (2020), pp. 513–544. DOI: 10.1007/s00211-020-01154-8. URL: https://doi.org/10.1007/s00211-020-01154-8 (cit. on p. 8).

[102] W. Jäger and S. Luckhaus. 'On explosions of solutions to a system of partial differential equations modelling chemotaxis'. In: *Trans. Amer. Math. Soc.* 329.2 (1992), pp. 819–824. DOI: 10.2307/2153966. URL: http://dx.doi.org/10.2307/2153966 (cit. on p. 5).

[103] R. Jordan, D. Kinderlehrer and F. Otto. 'The variational formulation of the Fokker-Planck equation'. In: *SIAM J. Math. Anal.* 29.1 (1998), pp. 1–17 (cit. on p. 5).

[104] L. Kantorovitch. 'On the translocation of masses'. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 37 (1942), pp. 199–201 (cit. on p. 3).

[105] T. Lachand-Robert and E. Oudet. 'Minimizing within convex bodies using a convex hull method'. In: *SIAM Journal on Optimization* 16.2 (Jan. 2005), pp. 368–379. URL: https://hal.archives-ouvertes.fr/hal-00385109 (cit. on p. 8).

[106] J.-M. Lasry and P.-L. Lions. 'Mean field games'. In: *Jpn. J. Math.* 2.1 (2007), pp. 229–260. DOI: 10.1007/s11537-007-0657-8. URL: http://dx.doi.org/10.1007/s11537-007-0657-8 (cit. on p. 5).

[107] L. Lebrat, F. de Gournay, J. Kahn and P. Weiss. 'Optimal Transport Approximation of 2-Dimensional Measures'. en. In: *SIAM Journal on Imaging Sciences* 12.2 (Jan. 2019), pp. 762–787. DOI: 10.1137/18M1193736. URL: https://epubs.siam.org/doi/10.1137/18M1193736 (visited on 25/03/2021) (cit. on p. 9).

[108] C. Léonard. 'A survey of the Schrödinger problem and some of its connections with optimal transport'. In: *Discrete Contin. Dyn. Syst.* 34.4 (2014), pp. 1533–1574. DOI: 10.3934/dcds.2014.34.1533. URL: http://dx.doi.org/10.3934/dcds.2014.34.1533 (cit. on p. 3).

[109] A. S. Lewis. 'Active sets, nonsmoothness, and sensitivity'. In: *SIAM Journal on Optimization* 13.3 (2003), pp. 702–725 (cit. on p. 7).

[110] B. Li, F. Habbal and M. Ortiz. 'Optimal transportation meshfree approximation schemes for Fluid and plastic Flows'. In: *Int. J. Numer. Meth. Engng 83:1541–579* 83 (2010), pp. 1541–1579 (cit. on p. 4).

[111] G. Loeper. 'A fully nonlinear version of the incompressible Euler equations: the semigeostrophic system'. In: *SIAM J. Math. Anal.* 38.3 (2006), 795–823 (electronic) (cit. on p. 4).

[112] G. Loeper and F. Rapetti. 'Numerical solution of the Monge-Ampére equation by a Newton's algorithm'. In: *C. R. Math. Acad. Sci. Paris* 340.4 (2005), pp. 319–324 (cit. on p. 3).

[113] S. G. Mallat. *A wavelet tour of signal processing*. Third. Elsevier/Academic Press, Amsterdam, 2009 (cit. on p. 6).

[114] B. Maury, A. Roudneff-Chupin and F. Santambrogio. 'A macroscopic crowd motion model of gradient flow type'. In: *Math. Models Methods Appl. Sci.* 20.10 (2010), pp. 1787–1821. DOI: 10.1142/S0218202510004799. URL: http://dx.doi.org/10.1142/S0218202510004799 (cit. on p. 5).

[115] Q. Mérigot. 'A multiscale approach to optimal transport'. In: *Computer Graphics Forum* 30.5 (2011), pp. 1583–1592 (cit. on p. 3).

[116] M. I. Miller, A. Trouvé and L. Younes. 'Geodesic Shooting for Computational Anatomy'. In: *Journal of Mathematical Imaging and Vision* 24.2 (Mar. 2006), pp. 209–228. URL: http://dx.doi.org/10.1007/s10851-005-3624-0 (cit. on p. 5).

[117] J.-M. Mirebeau. 'Adaptive, Anisotropic and Hierarchical cones of Discrete Convex functions'. In: *Numerische Mathematik* 132.4 (2016). 35 pages, 11 figures. (Second version fixes a small bug in Lemma 3.2. Modifications are anecdotic.), pp. 807–853. URL: https://hal.archives-ouvertes.fr/hal-00943096 (cit. on p. 8).

[118] E. Oudet and F. Santambrogio. 'A Modica-Mortola Approximation for Branched Transport and Applications'. en. In: *Archive for Rational Mechanics and Analysis* 201.1 (July 2011), pp. 115–142. DOI: 10.1007/s00205-011-0402-6. URL: http://link.springer.com/10.1007/s00205-011-0402-6 (visited on 06/01/2022) (cit. on p. 10).

[119] B. Pass. 'Uniqueness and Monge Solutions in the Multimarginal Optimal Transportation Problem'. In: *SIAM Journal on Mathematical Analysis* 43.6 (2011), pp. 2758–2775 (cit. on p. 4).

[120] B. Pass and N. Ghoussoub. 'Optimal transport: From moving soil to same-sex marriage'. In: *CMS Notes* 45 (2013), pp. 14–15 (cit. on p. 4).

[121] F.-P. Paty and M. Cuturi. *Regularized Optimal Transport is Ground Cost Adversarial*. 2020. arXiv: 2002.03967 [stat.ML] (cit. on p. 8).

[122] H. Raguet, J. Fadili and G. Peyré. 'A Generalized Forward-Backward Splitting'. In: *SIAM Journal on Imaging Sciences* 6.3 (2013), pp. 1199–1226. DOI: 10.1137/120872802. URL: http://hal.archives-ouvertes.fr/hal-00613637/ (cit. on p. 7).

[123] L. Rudin, S. Osher and E. Fatemi. 'Nonlinear total variation based noise removal algorithms'. In: *Physica D: Nonlinear Phenomena* 60.1 (1992), pp. 259–268. URL: http://dx.doi.org/10.1016/0167-2789(92)90242-F (cit. on p. 6).

[124] J. Solomon, F. de Goes, G. Peyré, M. Cuturi, A. Butscher, A. Nguyen, T. Du and L. Guibas. 'Convolutional Wasserstein Distances: Efficient Optimal Transportation on Geometric Domains'. In: *ACM Transaction on Graphics, Proc. SIGGRAPH'15* (2015). to appear (cit. on pp. 4, 5).

[125] R. Tibshirani. 'Regression shrinkage and selection via the Lasso'. In: *Journal of the Royal Statistical Society. Series B. Methodological* 58.1 (1996), pp. 267–288 (cit. on p. 6).

[126] R. Tovey and V. Duval. 'Dynamical Programming for off-the-grid dynamic Inverse Problems'. working paper or preprint. Dec. 2022. URL: https://inria.hal.science/hal-03500048 (cit. on p. 9).

[127] A. Vacher and F.-X. Vialard. ‘Parameter tuning and model selection in optimal transport with semi-dual Brenier formulation’. In: *NeurIPS*. New Orleans, France, 2022. URL: https://hal.science/hal-03475455 (cit. on p. 8).

[128] S. Vaiter, M. Golbabaee, J. Fadili and G. Peyré. ‘Model Selection with Piecewise Regular Gauges’. In: *Information and Inference* (2015). to appear. URL: http://hal.archives-ouvertes.fr/hal-00842603/ (cit. on p. 7).

[129] C. Villani. *Optimal transport*. Vol. 338. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Berlin: Springer-Verlag, 2009, pp. xxii+973. DOI: 10.1007/978-3-540-71050-9. URL: http://dx.doi.org/10.1007/978-3-540-71050-9 (cit. on p. 3).

[130] C. Villani. *Topics in optimal transportation*. Vol. 58. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+370 (cit. on p. 3).

[131] X.-J. Wang. ‘On the design of a reflector antenna. II’. In: *Calc. Var. Partial Differential Equations* 20.3 (2004), pp. 329–341. DOI: 10.1007/s00526-003-0239-4. URL: http://dx.doi.org/10.1007/s00526-003-0239-4 (cit. on p. 4).

[132] B. Wirth, L. Bar, M. Rumpf and G. Sapiro. ‘A continuum mechanical approach to geodesics in shape space’. In: *International Journal of Computer Vision* 93.3 (2011), pp. 293–318 (cit. on p. 5).

[133] J. Wright, Y. Ma, J. Mairal, G. Sapiro, T. S. Huang and S. Yan. ‘Sparse representation for computer vision and pattern recognition’. In: *Proceedings of the IEEE* 98.6 (2010), pp. 1031–1044 (cit. on p. 6).

[134] M. Yu. ‘Entropic Unbalanced Optimal Transport: Application to Full-Waveform Inversion and Numerical Illustration’. Theses. Université de Paris, Dec. 2021. URL: https://hal.inria.fr/tel-03512143 (cit. on p. 8).