

RESEARCH CENTRE

**Inria Saclay Centre at Université
Paris-Saclay**

IN PARTNERSHIP WITH:

CNRS, Université Paris-Saclay

2024

ACTIVITY REPORT

Project-Team

PARMA

Particle methods using Monge-Ampère

IN COLLABORATION WITH: Laboratoire de mathématiques d'Orsay de
l'Université de Paris-Sud (LMO)

DOMAIN

**Applied Mathematics, Computation and
Simulation**

THEME

**Optimization, machine learning and
statistical methods**

Inria

Contents

Project-Team PARMA	1
1 Team members, visitors, external collaborators	2
2 Overall objectives	3
2.1 Scientific and application context	3
2.1.1 Optimal transport	3
2.1.2 Computational optimal transport	3
3 Research program	4
3.1 Optimal transport: models, numerical analysis and solvers	4
3.1.1 Models	4
3.1.2 Numerical analysis	5
3.1.3 Solvers	5
3.2 OT-based particle methods for fluids mechanics	6
3.3 OT for inverse problems	8
3.3.1 OT as a data-fidelity term	8
3.3.2 Multi-marginal OT	9
3.4 Optimal transport for Social Sciences	10
3.5 Optimal transport for Cosmology	11
4 Application domains	11
4.1 Geography-Social science	11
4.2 Medicine	12
4.3 Optics	12
4.4 Crowd motion	12
4.5 Astrophysics	12
5 Social and environmental responsibility	12
5.1 Footprint of research activities	12
6 Highlights of the year	12
6.1 Awards	12
7 New software, platforms, open data	12
7.1 New platforms	13
7.1.1 SDOT	13
7.1.2 CroMoSim	13
7.1.3 PET KinetiX	13
7.1.4 GEOGRAM	13
8 New results	14
8.1 Ground state energy is not always convex in the number of electrons	14
8.2 Convergence rate of entropy-regularized multi-marginal optimal transport costs	14
8.3 A variational formulation of a Multi-Population Mean Field Games with non-local interactions	14
8.4 An ordinary differential equation for entropic optimal transport and its linearly constrained variants	15
8.5 Robust risk management via multi-marginal optimal transport	15
8.6 Gluing methods for quantitative stability of optimal transport maps	15
8.7 Crowd motion	16
8.8 Kinetic modeling for nuclear medicine	16
8.9 From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows	17
8.10 Metric extrapolation in the Wasserstein space	17
8.11 Regularity theory and geometry of unbalanced optimal transport	17

8.12 Monge Ampère gravity: from the large deviation principle to cosmological simulations through optimal transport	18
8.13 Faster is Slower effect for evacuation processes: a granular standpoint	18
9 Partnerships and cooperations	19
9.1 International initiatives	19
9.2 International research visitors	19
9.2.1 Visits of international scientists	19
9.3 National initiatives	20
9.4 Regional initiatives	20
10 Dissemination	20
10.1 Promoting scientific activities	20
10.1.1 Scientific events: organisation	20
10.1.2 Journal	21
10.1.3 Invited talks	21
10.1.4 Scientific expertise	23
10.1.5 Research administration	23
10.2 Teaching - Supervision - Juries	23
10.2.1 Teaching	23
10.2.2 Supervision	23
10.2.3 Juries	24
10.3 Popularization	24
10.3.1 Productions (articles, videos, podcasts, serious games, ...)	24
10.3.2 Participation in Live events	24
11 Scientific production	25
11.1 Major publications	25
11.2 Publications of the year	25
11.3 Cited publications	27

Project-Team PARMA

Creation of the Project-Team: 2024 January 01

Keywords

Computer sciences and digital sciences

- A6.1. – Methods in mathematical modeling
- A6.2. – Scientific computing, Numerical Analysis & Optimization
- A6.3. – Computation-data interaction
- A6.5. – Mathematical modeling for physical sciences
- A8.2. – Optimization
- A8.2.3. – Calculus of variations
- A8.12. – Optimal transport

Other research topics and application domains

- B2.6.3. – Biological Imaging
- B8.3. – Urbanism and urban planning

1 Team members, visitors, external collaborators

Research Scientists

- Thomas Gallouët [Team leader, INRIA, Researcher, HDR]
- Yann Brenier [CNRS, Emeritus]
- Bruno Levy [INRIA, Senior Researcher, HDR]

Faculty Members

- Maxime Laborde [Paris cité, Associate Professor, from Sep 2024]
- Bertrand Maury [UNIV PARIS SACLAY, Professor, from Sep 2024, HDR]
- Quentin Mérigot [UNIV PARIS SACLAY, Professor, HDR]
- Luca Nenna [UNIV PARIS SACLAY, Associate Professor, HDR]

Post-Doctoral Fellow

- Katharina Eichinger [UNIV PARIS SACLAY, from Oct 2024]

PhD Students

- Siwan Boufadene [UNIV GUSTAVE EIFFEL]
- Adrien Cances [UNIV PARIS SACLAY]
- Benjamin Capdeville [ENS PARIS-SACLAY, from Sep 2024]
- Quentin Giton [UNIV PARIS SACLAY, from Oct 2024]
- Erwan Stampfli [UNIV PARIS SACLAY]
- Christophe Vautier [UNIV PARIS SACLAY]

Technical Staff

- Sylvain Faure [CNRS, Engineer]
- Hugo Leclerc [CNRS, Engineer]

Administrative Assistant

- Aissatou-Sadio Diallo [INRIA]

External Collaborators

- Florent Besson [APHP, Université Paris-Saclay, PU-PH Service de Médecine Nucléaire - Imagerie Moléculaire Hôpitaux Universitaires Paris-Saclay AP-HP, site Bicêtre]
- Anna Korba [ENSAE]
- Roya Mohayaee [Institut d'Astrophysique de Paris, HDR]
- Patrick Poncet [Université Paris-Saclay, Géographe, Codirecteur de GéoMaths du PEPR Maths-VivES]

2 Overall objectives

The PARMA project focuses on the application of OT techniques to the numerical resolution of problems, where one wants to impose that a point cloud (representing either some data, or the result of a simulation) follows a certain probability distribution. This will for instance be done by using the Wasserstein distance as a penalization. We will especially focus on the numerical analysis and resolution relying on particle discretization (i.e. point clouds). PARMA will be organized around three main objectives:

- **Numerical analysis of OT and solvers (3.1):** The numerical analysis of optimal transport (convergence estimates, regularity properties of discrete solutions, a posteriori estimates) is still widely open, and needs to be further studied. Applications of this theoretical analysis include the development of efficient, robust and large-scale OT solvers, and a better mathematical understanding of the numerical schemes that we will develop in the next two objectives.
- **OT for fluid dynamics (3.3):** On the one hand it is known since the work of Brenier that optimal transport can be used to impose incompressibility in fluid dynamics [40]. This is illustrated in Figure 2.1.2. On the other hand many dissipative PDE/system of PDE can be interpreted as gradient flows in the Wasserstein space. Our aim within the PARMA team is to use these ideas to design, analyze and implement new particle schemes for more general evolution equations from fluid dynamics, biology and social sciences. Benefits of this approach include interface tracking and adaptive discretization; other benefits are discussed in Section 3.2.
- **OT for inverse problems (3.3):** Wasserstein distances have been used as a data fidelity term in various problems involving inverse problems where the signal generated by the model can be interpreted as a distribution of mass. For instance it is the case in seismic imaging [63], where it has been observed that OT distances tend to convexify the energy landscape, enlarge the basin of attraction of the optimal solution and make it possible to estimate the underground from a very crude guess (in comparison to what is obtained with a L^2 distance). Understanding both qualitatively and quantitatively this "convexification" property of OT is a long term goal of PARMA. We also plan to use Wasserstein distances as data fidelity terms in other applicative contexts.
- **OT for Social Sciences** Some preliminary attempts have been made to model the repartition of inhabitants in an urban area as a constrained optimization problem for a measure. In collaboration with geographers, we plan to extend those ideas to better account for social tendencies.

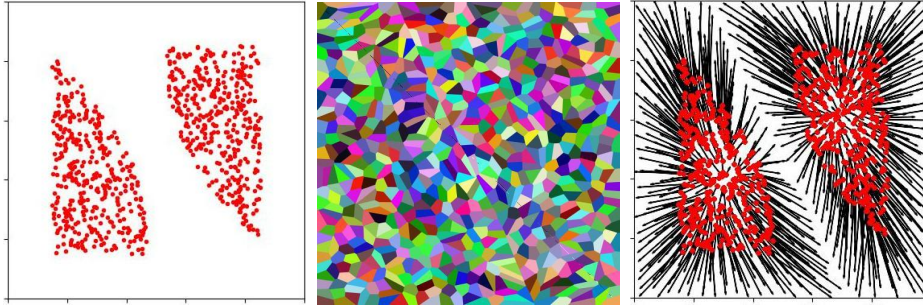
2.1 Scientific and application context

2.1.1 Optimal transport

The problem of optimal transport (OT), introduced by Gaspard Monge in 1871, was motivated by engineering applications: find the most economical way to transport a certain amount of sand from a quarry to a construction site. The modern theory of optimal transport has been initiated by Leonid Kantorovich in the 1940s, via a convex relaxation of Monge's problem. The Monge-Kantorovich theory can be used to define notions of distance between probability measures over the Euclidean space, often called Wasserstein distances. Unlike other notions — such as total variations, relative entropies, etc. — the Wasserstein distances fully encode the geometry of the underlying space, making them ideal to compare probability measures in a geometric context. This approach has been used and revisited by many authors from the 1980s, who connected it to various domains of mathematics, physics (fluid mechanics, quantum chemistry), economics (principal-agent problem, finance), mathematical learning (loss functions, statistics on point clouds), etc. Optimal transport is now more than ever a vivid topic, with more than 9200 articles published in 2021 containing the words "optimal transport" (Source: Google Scholar).

2.1.2 Computational optimal transport

In the 2000s, the theory of optimal transport was already mature and applied within mathematics, but also in theoretical physics or in economy. However, numerical applications were essentially limited to



1-dimensional problems because of prohibitive cost of existing algorithms, whose complexity is more than quadratic in the size of the data in dimension larger than two. Numerous numerical methods have been introduced to date, such as the Benamou-Brenier algorithm [34], but two in particular have changed this state of affairs:

- Entropic regularization of the optimal transport problem, which can be solved by Sinkhorn-Knopp's algorithm. This idea has been introduced in optimal transport by Alfred Galichon [49] (in Economy) and Marco Cuturi [47] (in Machine Learning).
- Semi-discrete formulation of optimal transport, introduced by Cullen in 1984 [46], which involves solving asymmetric problems between a probability density ρ and a probability measure with a finite support μ . The first effective implementation comes from the work of Q. Mérigot [59], using computational geometry techniques and a multiscale strategy.

Thanks to these two approaches, which have quite distinct domains of applicability, solving an optimal transport problem in small dimension with reasonable sized data (e.g. $N = 10^6$ in $d = 2, 3$) and for a "geometric" cost function has become a formality. Optimal transport can therefore be used, with the limitations set out above, as a brick within a more complex system. Applications requiring the resolution of optimal transport problems within an iterative algorithm include for example the resolution of inverse problems, machine learning, or the construction of numerical schemes for some evolution equations.

3 Research program

3.1 Optimal transport: models, numerical analysis and solvers

3.1.1 Models

Participants: Yann Brenier, Thomas Gallouët, Quentin Merigot, Bertrand Maury, Luca Nenna.

The team is interested in extending OT to multiple applications. The main ones are as follows:

- **Unbalanced and partial OT.** These metrics aim to compare measures with different masses [55]. They can be very pertinent for example when dealing with incomplete datum or PDE for which involve variation of mass (e.g. with a reaction term). These methods share a lot of property with classical OT [50]. A particular case of unbalanced optimal transport is the Moreau-Yosida regularization of the Wasserstein distance, which allows to define a notion of ϵ -entropy to a point cloud, and which plays an important role in numerical schemes for fluids dynamics (Section 3.2).
- **Linearized OT** is a way to lift up the Wasserstein space in a classical weighted L^2 Hilbert space, allowing one to apply the "linear" statistical toolbox to families of probability measures. Q. Mérigot and A. Delalande recently proved that this injection is bi-Hölder [60], thus showing a coarse relationship between this metric and the original OT metric. This work is specialised in the quadratic case, and calls for natural extensions to other OT models.

- **Unequal dimensional Optimal Transport.** Another natural extension of classical OT arises in Economy in the case in which one wants to compare two populations having a different amount of characteristics. This translates into an optimal transport problem between measures defined on space with different dimension. This kind of problem has been recently studied in [44, 58] and further developed in the framework of variational problems involving an OT term (for instance, the Cournot-Nash equilibria) in [64]. In particular we will focus on developing new metrics on the space of probability measures by means of unequal dimensional transport and study the link with a special instance of the linearised optimal transport LOT.
- **Gromov-Wasserstein.** This variant of OT has been introduced in order to take into account inner metric of the data structure. We aim to leverage the properties of this metric for the modelisation of geographic behaviour, for example we can try to use them in order to classify different cities. A long term objective here is to provide tools to help decision making in urban planning. This research is a part of the PEPR project "Maths vives" and done in collaboration with Geographers.

3.1.2 Numerical analysis

Participants: Thomas Gallouët, Quentin Merigot, Luca Nenna.

Convergence speed. The numerical analysis of optimal transport is still in its infancy. The convergence of the solutions of the discretized optimal transport problem towards solutions of the continuous problem is usually obtained by a compactness argument, which is therefore not quantitative. For many applications, it is necessary to have a more explicit control of the distance between the optimal transport maps for different data (e.g. discrete and continuous), depending on the distance between the data. This problem is difficult even if we restrict ourselves to optimal transport for the quadratic cost (mainly because the Monge-Ampère equations appearing in transport optimal is not strongly elliptic). Recent results have been obtained by members of the team, in the quadratic case [60], but many questions are still open: extension to other costs, improved exponents, a posteriori analysis, etc. We also expect applications to the design and analysis of fast multi-scale algorithms for optimal transport.

Discrete regularity For many of our targeted applications, it is important to understand quantitatively the regularity of solution of semi-discrete optimal transport problems. The question is to find a discrete analogue of the regularity theory for the optimal transport (continuity method, or \mathcal{C}^1 regularity *à la* Caffarelli), which would imply for instance that when the point cloud is close to a probability density, then the Laguerre cells are not too anisotropic. Should this be true, it would also have consequences in the analysis of the particle schemes for fluid dynamics proposed in §3.2, but would also provide a better understanding of the behaviour of Newton's algorithm for semi-discrete OT.

3.1.3 Solvers

Participants: Sylvain Faure, Hugo Leclerc, Bruno Levy, Quentin Merigot, Luca Nenna.

Algorithms Despite the recent progress on the algorithmic aspects of optimal transport, many questions are still not understood: most of the existing complexity analysis are in a worst-case setting, and thus do not explain *at all* the practical good behaviour of OT solvers. More generally, we expect that the progress in the numerical analysis of OT will lead to the design of more efficient solvers, using multi-scale and preconditioning strategies (in semi-discrete OT) or ϵ -scaling techniques (in the context of entropy-regularized OT).

Implementation of a semi-discrete solver Technical work has been done to be able to generate power diagrams and associated integrals in fast and memory efficient way. Power diagrams are at the center of the calculations to obtain Kantorovich potentials for optimal transport in semi-discrete settings. Among other things, the aim is to be able to use the capabilities of GPUs and SIMD instructions on CPUs (parallel instructions), and then to be able to scale up to cluster-based calculations. The first results are very encouraging since some cases show significantly lower computation times, naturally exploiting the parallelism of contemporary architectures, and above all, for an almost negligible memory occupation (compared to what was previously required).

These properties allow us to seriously consider the development of new classes of iterative algorithms adapted to different semi-discrete OT problems, which would be very fast and inexpensive in terms of memory occupation. Some first proposals to improve the robustness or the convergence speed (adapted preconditioners) have shown promising results. Moreover, extensions to larger numbers of dimensions have started and will be followed up in the context of the project. All generic developments are deposited under open source licenses, associated with rigorous documentation and testing procedures.

Milestones 4 years Objective: An efficient and user friendly Semi-Discrete OT solver is yet to exist. This is the objective of Sdot (solver written in C++) and PySdot (interface with Python), developed by H. Leclerc. It will have to be accessible for non-specialists and non mathematicians users and able to solve of wide range of problems. This software is open source, and is currently used by some colleagues but the API is not stable yet. To develop our strategy on these questions we hope to benefit from the experience and expertise of Inria.

- **Task (GEOT-NP)** in the context of geography (urban planning), we propose to develop conceptual and numerical tools to analyse and compare different areas represented as measured metric spaces (population densities and public transportation network). We aim in particular at developing an adapted metric, in the spirit of Gromov Wassertsin distance, which would incorporate information on the paths which are really used (based on origin destination matrices), which is not the case for the existing GW distance.
- **Task (API-V1)** improve the Python API to make it more pythonic, and to allow an easy combination with existing machine learning libraries (Scikit-Learn, PyTorch, etc.), especially allowing seamless use of automatic differentiation.
- **Task (API-D)** Write a detailed documentation for Task **(API-V1)**.
- **Task (API-V2)** Create bindings/interface for other languages (julia, R).

8-12 years Objective :

- **Task (GEOT-V)** Valorization of the interactions developed in Task **(GEOT-NP)** .
- **Task (API-VN)** Develop a very fast parallel SD-solver for high resolution problems in 2D/3D, exploiting the capabilities of modern hardware (GPU, SIMD).

3.2 OT-based particle methods for fluids mechanics

Participants: Yann Brenier, Sylvain Faure, Thomas Gallouët, Bruno Levy, Hugo Leclerc, Bertrand Maury, Quentin Merigot.

A number of evolution problems, appearing in physics (porous media, incompressible Euler, isentropic Euler), in biology (chemotaxis) or human sciences (crowd movements) can formally be described as the evolution of a large system of particles. We consider the lagrangian variant of these problems, where one is looking for the evolution of a displacement field by a dissipative or a conservative evolution:

- Dissipative equations are associated with the notion of gradient flows in the Wasserstein space. This idea was originally proposed by Otto in the late 90s [54, 65] and systematically studied in the seminal book by Ambrosio-Gigli-Savare [33]. Many equations enter this framework, modeling porous media [66], crowd movements, chemotaxy, etc. The gradient flow formulation is used for several purpose: analysis (uniqueness, convergence to equilibrium), modeling (rate of dissipation of some energies) and construction of numerical schemes.
- Conservative equations are associated with geometric methods for hydrodynamics, starting from Arnold's interpretation of the incompressible inviscid Euler's equations as geodesics in the space of volume-preserving diffeomorphisms. This has been revisited, in the context of optimal transport, by Brenier: by his polar factorization theorem, the computation of the distance between a map from the domain to itself and the space of volume-preserving diffeomorphisms is equivalent to an OT problem.

In combination with semi-discrete optimal transport, these ideas lead to Lagrangian numerical schemes for Euler's equation for incompressible and compressible fluids, and for many gradient flows with a suitable energy functional [52, 51]. Many other evolution problems can be formulated in this framework, perhaps with some adaptation (fluid-structure interactions, problems involving surface tension, and/or interaction between particles, etc.). Our goal within the PARMA project is to derive systematically particle discretizations of these forms, using optimal transport or variants thereof (e.g. partial or unbalanced optimal transport). In particular, we will adapt this strategy to the crowd motion models proposed by B. Maury [57] and to pressureless Euler equation [56], which both rely on partial optimal transport. This approach offers many advantages over existing numerical methods:

- Semi-discrete OT automatically associates a moving mesh to the particle discretization, allowing one to track precisely the interfaces between particles, e.g. between two phases of a fluid (see Fig. 2.1.2).
- These discretizations respect the structure of the equations, which is therefore recovered at the discrete level some properties of the equation, such as mass preservation and positivity, or convergence to the equilibrium estimates coming from the gradient flow structure.
- These particle approach allows us to simulate equations with an explosive behaviour, which would be difficult to describe finely using fixed-mesh methods. The OT approach makes it easy to refine/coarsen the solution in a principled way, simply by looking at the shape of Laguerre cells.

Our approach bears some similarity with smoothed particle hydrodynamics (SPH) discretizations, where the interaction forces amongst the particles are computed by reconstructing the fluid density through convolution with a fixed kernel, and which have been widely used in the context of the discretization of fluid models. However, these methods are prone to instability and are difficult to analyze from a mathematical viewpoints: very few convergence results exist, and recent works show that the choice of the kernel is both non-trivial and crucial to obtain convergence.

Milestones 4 years Objective:

- **Task (INS-P)** Incompressible Navier-Stokes (INS). We already know how to implement a SD-OT scheme for INS. The proof is yet to be done. In order to adapt our proof done for Incompressible Euler one need to deal with the dissipation term. This term is not easy to handle our strategy will be to desymetrize the remaining non quadratic term.
- **Task (INSF-PN)** The following step will be to add more physics to this model, for example surface tension.
- **Task (ORD2-PN)** We aim to develop a space-time order 2 numerical scheme for a wide class of gradient flow. This will be done thanks to a metric formulation of BDF-2 scheme and appropriate notion of Wasserstein extrapolation studied in Task (**Ext-PN**).

- **Task (EDGF-PN)** We study the gradient flow where the interaction energy is given by the distance and especially the long time behaviour of this motion. This problem is non-convex but seems to converge to a steady state anyway. Here we want to leverage some harmonic analysis properties. This task is the subject of the Phd of Siwan Boufadene.
- **Task Muskat-P** In his Phd thesis Erwan Stämpfli is studying the Muskat problem. In particular using modulated energy methods he proves the convergence of multiphase flows towards the Muskat problem in dimension 1.

8-12 years Objective :

- **Task (SG-PN)** The semi-geostrophic model is related to optimal transport. SD models are well adapted in order to approximate its solutions. Proving the convergence of the scheme is more challenging especially in the 2D approximation. A first step in this direction is the Task (**Quant-P**) describe below.
- **Task (Inter-PN)** Another driving direction is to add interactions term in these model. The relative entropy method is well adapted for controlling these term and works in the mean field limit community.

3.3 OT for inverse problems

3.3.1 OT as a data-fidelity term

Participants: Thomas Gallouët, Quentin Merigot, Sylvain Faure

In this context optimal transport is used as a way to evaluate the distance from a measure to given data. Let us mention two class of problem following this approach.

- **Constraint minimisation problem.** The first one is given by the minimisation of a functional under some fidelity data attachment given by an OT problem. The construction of numerical approximations of Brenier's generalized solutions for incompressible Euler, done by Q. Mérigot and J. M. Mirebeau [61], is a typical example of such problems where the kinetic energy of a large number of trajectories are minimized under incompressibility constraints enforced with OT. Changing the minimization of the kinetic energy of the velocity field by the square of the acceleration leads to the definition of Wasserstein splines as introduced by J.D. Benamou, T. Gallouët and F.X. Vialard [38], providing a way to construct smooth interpolations between probability measures. We will pursue the work in these directions. This type of approaches is also popular in imaging, with contributions from K. Bredies, B. Schmitzer, H. Lavenant, etc.
- **Parametric minimization.** We are interested in another class of problem where one wants to minimize the Wasserstein distance between the data (ρ) and a measure (μ_θ) obtained for example either through a parametric model or by a more complex simulation. The problems can be written as follows, where θ lives in some parameter space:

$$\min_{\theta} W_p^p(\mu_\theta, \rho). \quad (1)$$

A typical problem falling into this category is the Wasserstein generative adversarial networks (WGAN), which seeks to construct a model able to generate data statistically compatible with a dataset. We have several ongoing collaborations around problems of the form (1). First, with astrophysicists from the Observatoire de Paris (R. Mohayaee), around the early universe reconstruction problem [39] and now with more general problems which can be put under the form (1). Another ongoing collaboration on this type of problem is the reconstruction of a model of the underground using full waveform inversion techniques, where optimal transport has already proven to be a valuable tool to avoid cycle skipping [62]. Finally, non-imaging optics problems are solved using the formulation (1), and is the object of a maturation project with the Linksium SATT (ANIDOLIX project, with J.B. Keck, Q. Mérigot and B. Thibert).

One of the main difficulty of some of these problems is that their Eulerian formulation, while convex can be very high dimensional. The particle (or Lagrangian) discretizations are lower dimensional but non-convex. However, we hope to leverage some traces of convexity properties present at the Eulerian level in order to prove quantitative estimates on the local minimizers as done in [60].

3.3.2 Multi-marginal OT

Participants: Quentin Merigot, Luca Nenna.

Multi-marginal optimal transport is a challenging variant of OT, where one seeks a coupling between more than two probability measures minimizing some cost functionals. These kind of problems arise naturally in physics such as the computation of electronic density in quantum chemistry [42, 45] (Density Functional Theory) or in fluid dynamics, Brenier's generalized solutions of incompressible Euler equations [41], as well as computation of barycenters in the Wasserstein space [30] (with its many applications to machine learning and imaging analysis), and other applications in economics (matching for teams [43]) and risk estimation, as we detail below. Understanding the structure, regularity and sparsity properties (namely the existence of Monge minimizers) of optimal plans for multi-marginal transport problems is a very active and challenging area of research. Fast numerical solvers yet are still to be found to address these typically very high-dimensional problems. For instance, a crude discretization of each of a multimarginal OT problems with 6 marginals and using 100 Dirac masses per marginal would yield a convex optimization problem with 10^{12} unknowns! This makes this convex formulation of the problem practically intractable.

Application: Risk estimation We want to study the modeling of risk in an industrial setting, where the distribution of several risk factors is known but the joint probability distribution is unknown. For safety reasons, one sometimes needs to have a pessimistic estimation of the risk, i.e. finding a coupling between the distribution of risk factors that maximize a certain measure of risk. An example of such a problems for evaluating the risk of an industrial facility close to a river and protected by a dyke can be found in [53]: one knows how to evaluate a risk (such as the level of water in the river, compared to the height of the dyke) depending on a few variables (e.g. river width, maximal annual flowrate, etc.). What is unknown however, is the coupling between these variables. Natural questions in the context of risk estimation are the following: (1) What is the coupling (i.e. the correlation) between the variables that yields the worst-case scenario, e.g. the highest mean risk? (2) Which coupling gives the highest mean risk among the e.g. 1% riskiest events? Answering these questions leads to an high dimensional multi-marginal (or partial multi-marginal) optimal transport:

$$\min_{\gamma} \int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_\ell}} c(x_1, \dots, x_\ell) d\gamma(x_1, \dots, x_\ell). \quad (2)$$

In the above problem, c is a cost function, and the minimum is taken over $\gamma \in \mathcal{M}^+(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_\ell})$ a non-negative measures of mass α over $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_\ell}$, whose i th marginal $\Pi_{i\#}\gamma$ (that is the projection of γ on \mathbb{R}^{d_i}) is upper bounded by a prescribed probability measure μ_i on \mathbb{R}^{d_i} . The case $\alpha = 1$ corresponds to the multi-marginal optimal transport problem.

Application: density functional theory Another interesting application of multi-marginal transport arises quite naturally in the framework of *Density Functional Theory*: it has been recently realized by mathematicians [45, 42] that the central problem of finding the lowest Coulomb energy of N -particle probabilities at given first marginal (a.k.a. charge density) ρ is indeed *multi-marginal optimal transport problem* similar to (2) where the cost is now the Coulomb potential, $\mu_i = \rho$ for all i and $\alpha = 1$.

Objectives: The main objective on the theoretical side will be then to characterize the dimension of the support of the solution of (2) in general context. This information could allow to further improve some existing numerical methods such as the Sinkhorn algorithm based on the entropic regularization of

Optimal Transport [35, 36, 37] and the ones arising from the approximation of optimal transport problems with marginal moments constraints [32, 31].

Milestones 4 years Objective:

- **Task (Ext-PN)** A task falling in the category is the definition of a Wasserstein geodesic extrapolation. This is not obvious due to the presence of possible shocks appearing while extending a Wasserstein geodesic. We will propose different notions of extrapolation and numerical scheme in order to compute them. The SD approach seems particularly promising to tackle the metric extrapolation. This discretization leads to a non-convex approximation of a convex problem.
- **Task (Quant-P)** In this task we investigate the convergence of the gradient flow for the quantization problem. We use here a relative entropy method and think of this task as a toy model for more intricate dynamics such as Tasks **(Quant-P)** and **(SG-PN)**.
- **Task (BQuant-P)** This task is similar as **(Quant-P)** but this time we aim to quantize the barycenter of a fixed number of measures. This generalization is not straightforward since the dynamic of the energy is less well understood.
- **Task (Risk-Pa)** Characterize **the dimension** of the support of the solution of MMOT in a general context. We plan to rely on and to extend the work of B. Pass [67], which gives upper bounds on this dimension under technical assumptions on the cost.
- **(Risk-Pb)** Study the behaviour of a simple **particle discretization** of the problem, where the solution is a sum of a finite number of Dirac masses whose positions are free to move. The constraints could be enforced via penalization, using optimal transport data attachment terms. This discretization makes the discrete problem non-convex, but one could hope to study the convergence of global minimizers. Notice that a coarser discretization, thanks to a better understanding of the structure of the solution, will make the numerical method competitive with respect to the actual ones: in the entropic case the dimension of the support of the solution is naturally high dimensional since the regularization effect, meaning that a finer discretization is needed.
- **Task (Risk-N)** Implement this algorithm and experiment it numerically on some applications as the risk estimation problem presented in [53]. These three **Risk** tasks are the core of A. Cancès Phd thesis.
- **Task (MMOT-DFT-N)** In quantum chemistry, the MMOT problem models the electron-electron repulsion, but numerical simulations to compute the solution to MMOT can be afforded only for a very small number of electrons/marginals. Here we aim at generalize the numerical methods to the case of grand canonical optimal transport introduced in [48], a useful generalization of MMOT which let us to decompose a molecule in sub-domain (with a less number of electrons/marginals) and compute the electronic configuration in each of them.

8-12 years Objective :

- **Task (MMOT-Prosp)** Since multi-marginal optimal transport arises in many domains, and especially in **machine learning**, **imaging science** and **computational quantum chemistry**, we expect that the theoretical and numerical tools developed during the project will find many other applications and help us to further improve interactions with other domain as the already existing ones with chemists (e.g. L. Nenna already works with Paola Gori-Giorgi, a renowned theoretical chemist based in Amsterdam) or with researchers from EDF working on risk measures.

3.4 Optimal transport for Social Sciences

Participants: Sylvain Faure, Bertrand Maury.

In the context of the Géomaths project (PEPR Maths VivES), some research axes are connected to Optimal Transportation. In particular, we aim at analyzing (population) density distributions in terms of Wasserstein distance to the uniform density. The knowledge of the optimal transport plan will give some information concerning the most predominant scales of the zone, and possibly predominant directions. Another research track follows some preliminary steps taken by F. Santambrogio and G. Buttazzo. It consists in formulating the question of population settlement in terms of optimal transportation. A population represented by a density aims at optimizing its utility by minimizing a functional which encodes various tendencies. We shall develop an incremental version of this approach, in order to account for the fact that a city does not develop from scratch, but rather by successive phases of migration and construction. We also plan to explore the possibility to account for populations with various tendencies (people are not interchangeable).

3.5 Optimal transport for Cosmology

Participants: Quentin Leclerc, Bruno Levy, Quentin Merigot.

Modern Cosmology is still confronted with two fundamental questions:

1. the rotation curves of galaxies imply *dark matter*, the nature of which is yet to be understood;
2. explaining the accelerated expansion of the Universe requires a mysterious *dark energy*.

The situation today seems favorable: increasingly accurate acquisition data is available (DESI, Euclid, Gaia ...), and computer power has continued to make considerable progress. To gain understanding on the evolution of the Universe throughout its history, one can try to reconstruct the movement of all galaxies by solving an inverse problem (Peeble's action method). This problem was characterized as an optimal transport problem by Firsich, Matarrese, Sobolevskii and Mohayaee, and solved numerically using Bertsekas's algorithm for discrete optimal transport on problems of moderate size (hundred thousands to millions Dirac masses).

We develop new semi-discrete optimal transport algorithms, as well as computational methods to construct the involved Laguerre diagrams, in order to scale up to sizes of $10^6 - 10^9$ Dirac masses. This makes it possible to reconstruct the trajectories of galaxies, and enhance a subtle signal in the initial condition (Baryonic Acoustic Oscillations), characteristic of the early history of the Universe. This also makes it possible to conduct large scale numerical simulations for modified laws of gravity (Brenier-Monge-Ampère [15]).

4 Application domains

4.1 Geography-Social science

In the context of the Géomaths project (PEPR Maths VivES), some research axes are connected to Optimal Transportation. In particular, we aim at analyzing (population) density distributions in terms of Wasserstein distance to the uniform density. The knowledge of the optimal transport plan will give some information concerning the most predominant scales of the zone, and possibly predominant directions. Another research track follows some preliminary steps taken by F. Santambrogio and G. Buttazzo. It consists in formulating the question of population settlement in terms of optimal transportation. A population represented by a density aims at optimizing its utility by minimizing a functional which encodes various tendencies. We shall develop an incremental version of this approach, in order to account for the fact that a city does not develop from scratch, but rather by successive phases of migration and construction. We also plan to explore the possibility to account for populations with various tendencies (people are not interchangeable).

4.2 Medicine

Positron emission tomography is a common imaging technique, a medical scintillography technique used in nuclear medicine. A radiopharmaceutical - a radioisotope attached to a drug - is injected into the body as a tracer. When the radiopharmaceutical undergoes beta plus decay, a positron is emitted, and when the positron interacts with an ordinary electron, the two particles annihilate and two gamma rays are emitted in opposite directions. In the future, we plan to study the reconstruction of 3D PET-scan images from raw data (listmod i.e. positron emission events) using an optimal transport approach.

4.3 Optics

Anidolic or non-imaging optics is a branch of optics that aims to design devices (lenses or mirrors) that reflect or refract light emitted from a source toward a target, following a prescribed intensity distribution and support geometry. Typically, anidolic optics problems where the source is ideal (point-like or collimated) and the target is in the far field are formulated as optimal transport problems or Monge-Ampère equations. On the other hand, problems where the target is in the near field are formulated as a generated Jacobian equation. Semi-discrete optimal transport solvers are especially well suited for these family of problems.

4.4 Crowd motion

If you have people-counting sensors that measure the flow of people into and out of a location, you can estimate the time people spend there, using an optimal transport approach. Two applications have already been realized: the calculation of waiting times for company restaurants, and the calculation of attendance times for visitors or worshippers at Notre-Dame de Paris.

4.5 Astrophysics

Some inverse problems in astrophysics have a natural connection with optimal transport theory, such as Peeble's action problem, that reconstructs the motion of galaxies back in time. It is stated as an action minimization problem, equivalent to an optimal transport problem (Firsch, Matarrese, Sobolevskii, Mohayaee, Brenier). Efficient solvers for optimal transport open the door to new computational tools, that will make it possible to extract knowledge about the history of the Universe from observational data.

5 Social and environmental responsibility

5.1 Footprint of research activities

the team always prioritizes train travel for trips within Europe, and a large majority of its members refuse to travel overseas at all of for short periods. In any case within the spirit of the Labo 1.5 project.

6 Highlights of the year

6.1 Awards

Frontier of Science Award (L. Nenna)

Election of B. Maury at Académie des Sciences

7 New software, platforms, open data

You can add a header for the 'New software, platforms, open data' section

7.1 New platforms

7.1.1 SDOT

The INRIA team has enabled us to develop a new computing platform for optimal transportation in the semi-discrete setting (**sdot**). This platform is unique in the sense that the work involved in making optimal transportation accessible to the widest possible communities is currently almost exclusively focused in the discrete-discrete setting (e.g. with packages like POT or Geomloss which are excellent but are specialized toward discrete-discrete problems), which is not the case of Sdot.

We previously had a package named pysdot (**pysdot**) which served as a test bed for the API and the implementation choices. The design of Sdot was an opportunity to set things straight, with a much clearer and more flexible API, and to catch some fantastic openings for execution speed and memory usage (parallelism, GPUs, etc...).

We started the documentation and the communication processes to ensure the widest adoption. This package is already used outside the team and the laboratory, but given the possibilities it offers, we hope it will reach a significantly larger number of people

For the near future, we would like to add compatibility to other frameworks (like PyTorch, ...) and other languages.

7.1.2 CroMoSim

Participants: Sylvain Faure, Bertrand Maury.

The initial aim of cromosim was to enable reproducibility of the results present in the book “Crowds in equations: an introduction to the microscopic modeling of crowds” published in 2018 by Bertrand Maury and Sylvain Faure, and to provide a starting point for people wishing to model crowd movements. The aim was to propose several mathematical models, as well as common tools enabling the use of complex building plans. The software is currently available as a package written in Python, which can be installed via PyPI (pip). The complete code is available on Github (**cromosim**), and several examples of use are available. Cromosim enables calculations to be made on "real" building plans, by defining a color code characterizing the destinations of individuals, walls or furniture... Several microscopic mathematical models of crowd movements can thus be used: social force or granular-inspired models, cellular automata, compartment models. The initial target audience was mainly students and researchers, as all the "ingredients" of the models are accessible and modifiable. The commercial softwares used today, for example by engineering firms, are complex and includes numerous parameters that can influence the calculation of an evacuation time. What's more, they are essentially based on Helbing's microscopic model (social forces) or on cellular automata (individuals evolving in boxes). As a result, there is an interest in a non-black-box open-source software package, optimized for simulating dense crowds in real geometry, with well-documented parameterization and enabling results obtained with several mathematical models to be compared. In the coming years, we hope to build up a community of users by developing a web application around this python module.

7.1.3 PET KinetiX

Participants: Florent Besson, Sylvain Faure.

PET KinetiX is a software package for nuclear physicists and clinicians. It has been partly developed thanks to funding from CNRS Innovation (Prematuration and RISE programs) and is currently benefiting from a BPI BFTLaB grant to help prepare its commercialization. It is currently being tested in four hospital centers. The software's functions and initial results are described in [10].

7.1.4 GEOGRAM

Participants: Hugo Leclerc, Bruno Levy, Quentin Merigot.

GEOGRAM is a programming library with geometric algorithms. It has geometry-processing functionalities such as reconstruction, remeshing, parameterization, intersections, constructive solid geometry.

It has low-level functionalities such as Delaunay triangulation in 2D and in 3D, highly efficient parallel Delaunay triangulation in 3D used for cosmology, exact predicates, and efficient numerical solvers.

It has **efficient solvers for semi-discrete optimal transport in 2D and in 3D**.

8 New results

8.1 Ground state energy is not always convex in the number of electrons

Participants: Simone Di Marino, Mathieu Lewin, Luca Nenna

In this work [4] we provide the first counterexample showing that the ground state energy of atoms/molecules in an external Coulomb potential is not always a convex function of the number of electrons. This convexity has been conjectured for decades and plays a major role in quantum chemistry. Our counterexample involves six nuclei with small fractional charges placed far apart. The ground state energy of 3 electrons is shown to be higher than the average of the energies for 2 and 4 electrons. We also show that the nuclei can bind 2 or 4 electrons, but not 3. This article raises the question of whether the energy convexity really holds for all possible molecules (with nuclei of integer charge).

8.2 Convergence rate of entropy-regularized multi-marginal optimal transport costs

Participants: Luca Nenna, Paul Pegon

In this work [8] investigate the convergence rate of multi-marginal optimal transport costs that are regularized with the Boltzmann-Shannon entropy, as the noise parameter ε tends to 0. We establish lower and upper bounds on the difference with the unregularized cost of the form $C\varepsilon \log(1/\varepsilon) + O(\varepsilon)$ for some explicit dimensional constants C depending on the marginals and on the ground cost, but not on the optimal transport plans themselves. Upper bounds are obtained for Lipschitz costs or locally semi-concave costs for a finer estimate, and lower bounds for C^2 costs satisfying some signature condition on the mixed second derivatives that may include degenerate costs, thus generalizing results previously in the two marginals case and for non-degenerate costs. We obtain in particular matching bounds in some typical situations where the optimal plan is deterministic.

8.3 A variational formulation of a Multi-Population Mean Field Games with non-local interactions

Participants: Luigi De Pascale, Luca Nenna

In this work [27] we propose a MFG model with quadratic Hamiltonian involving N populations. This results in a system of N Hamilton-Jacobi-Bellman and N Fokker-Planck equations with non-local interactions. As in the classical case we introduce an Eulerian variational formulation which, despite the non convexity of the interaction, still gives a weak solution to the MFG model. The problem can

be reformulated in Lagrangian terms and solved numerically by a Sinkhorn-like scheme. We present numerical results based on this approach, these simulations exhibit different behaviours depending on the nature (repulsive or attractive) of the non-local interaction.

8.4 An ordinary differential equation for entropic optimal transport and its linearly constrained variants

Participants: Joshua Hiew, Luca Nenna, Brendan Pass

In [25] we characterize the solution to the entropically regularized optimal transport problem by a well-posed ordinary differential equation (ODE). Our approach works for discrete marginals and general cost functions, and in addition to two marginal problems, applies to multi-marginal problems and those with additional linear constraints. Solving the ODE gives a new numerical method to solve the optimal transport problem, which has the advantage of yielding the solution for all intermediate values of the ODE parameter (which is equivalent to the usual regularization parameter). We illustrate this method with several numerical simulations. The formulation of the ODE also allows one to compute derivatives of the optimal cost when the ODE parameter is 0, corresponding to the fully regularized limit problem in which only the entropy is minimized.

8.5 Robust risk management via multi-marginal optimal transport

Participants: Hamza Ennaji, Quentin Mérigot, Luca Nenna, Brendan Pass.

In [13] we study the problem of maximizing a spectral risk measure of a given output function which depends on several underlying variables, whose individual distributions are known but whose joint distribution is not. We establish and exploit an equivalence between this problem and a multi-marginal optimal transport problem. We use this reformulation to establish explicit, closed form solutions when the underlying variables are one dimensional, for a large class of output functions. For higher dimensional underlying variables, we identify conditions on the output function and marginal distributions under which solutions concentrate on graphs over the first variable and are unique.

8.6 Gluing methods for quantitative stability of optimal transport maps

Participants: Cyril Letrouit, Quentin Mérigot.

In [26], we establish quantitative stability bounds for the quadratic optimal transport map T_μ between a fixed probability density ρ and a probability measure μ on \mathbb{R}^d . Under general assumptions on ρ , we prove that the map $\mu \mapsto T_\mu$ is bi-Hölder continuous, with dimension-free Hölder exponents. The linearized optimal transport metric is therefore bi-Hölder equivalent to the 2-Wasserstein distance, which justifies its use in applications.

We show this property in the following cases: (i) for any log-concave density ρ with full support in \mathbb{R}^d , and any log-bounded perturbation thereof; (ii) for ρ bounded away from 0 and $+\infty$ on a John domain (e.g., on a bounded Lipschitz domain), while the only previously known result of this type assumed convexity of the domain; (iii) for some important families of probability densities on bounded domains which decay or blow-up polynomially near the boundary. Concerning the sharpness of point (ii), we also provide examples of non-John domains for which the Brenier potentials do not satisfy any Hölder stability estimate.

Our proofs rely on local variance inequalities for the Brenier potentials in small convex subsets of the support of ρ , which are glued together to deduce a global variance inequality. This gluing argument is

based on two different strategies of independent interest: one of them leverages the properties of the Whitney decomposition in bounded domains, the other one relies on spectral graph theory.

8.7 Crowd motion

Participants: Sylvain Faure, Bertrand Maury.

In [9] we study the so-called Faster is Slower (FIS) effect which is observed in some particular real-life or experimental situations. In the context of an evacuation process, it expresses that increasing the speed (or, more generally, the competitiveness) of individuals may induce a reduction of the flow through the exit door. We propose here a parameter-free model to reproduce and investigate this effect (more precisely its backward “Slower is Faster” equivalent). In spite of its non-smooth character, which makes it difficult to analyze, this granular approach is based on very basic ingredients in terms of behavior. In its native, purely asocial version, individuals are represented by hard-discs, each of which has a desired velocity, and the actual velocity is built as the projection of this field on the set of admissible velocities (which respect the non-overlapping constraints). We implement the slower effect by introducing here an extra step to account for the fact that individuals refrain from pushing, and therefore tend to reduce their desired velocity accounting for the velocities of people upfront. The present paper has two objectives: establish the relevance of this model by showing that it satisfactorily reproduces various empirical effects in highly crowded evacuations with various levels of competitiveness, and explore how it can be implemented to recover and explain the FIS effect. In this spirit, we confront this Inhibition-Based (IB) model to experimental data, focusing on the Faster is Slower effect. We show in particular that this approach makes it possible to accurately recover the effect of competitiveness upon power-law distributions of time lapses which have been experimentally observed. We also study the effect of mixed behaviors, by introducing a two-population model using both approaches. We investigate in particular the effect upon evacuation efficiency of the ratio between competitive agents and non-competitive ones. In a similar context, we investigate the role of an obstacle placed upstream the exit upon evacuation efficiency.

8.8 Kinetic modeling for nuclear medicine

Participants: Florent Besson, Sylvain Faure.

The purpose of [10] was to announce the birth of PET KinetiX software to the nuclear medicine community. Mathematically, the aim is to identify the parameters of several biological models at each point of the image (several millions), the difficulty being to perform these calculations quickly enough for clinical use. Optimal Transport is not part of this work, but it will be of interest in the future: for reconstructing the data used, and for comparing the results obtained for different groups of patients. This work therefore concerns the kinetic modeling which represents the ultimate foundations of Positron Emission Tomography (PET) quantitative imaging, a unique opportunity to better characterize the diseases or prevent the reduction of drugs development. Primarily designed for research, parametric imaging based on PET kinetic modeling may become a reality in future clinical practice, enhanced by the technical abilities of the latest generation of commercially available PET systems. In the era of precision medicine, such paradigm shift should be promoted, regardless of the PET system. In order to anticipate and stimulate this emerging clinical paradigm shift, we developed a constructor-independent software package, called PET KinetiX, allowing a faster and easier computation of parametric images from any 4D PET DICOM series, at the whole field of view level. The PET KinetiX package is currently a plug-in for Osirix DICOM viewer. The package provides a suite of five PET kinetic models: Patlak, Logan, 1-tissue compartment model, 2-tissue compartment model, and first pass blood flow. After uploading the 4D-PET DICOM series into Osirix, the image processing requires very few steps: the choice of the kinetic model and the definition of an input function. After a 2-min process, the PET parametric and error maps of the chosen model are automatically estimated voxel-wise and written in DICOM format. The software

benefits from the graphical user interface of Osirix, making it user-friendly. Compared to PMOD-PKIN (version 4.4) on twelve 18F-FDG PET dynamic datasets, PET KinetiX provided an absolute bias of 0.1% (0.05–0.25) and 5.8% (3.3–12.3) for KiPatlak and Ki2TCM, respectively. Several clinical research illustrative cases acquired on different hybrid PET systems (standard or extended axial fields of view, PET/CT, and PET/MRI), with different acquisition schemes (single-bed single-pass or multi-bed multipass), are also provided. PET KinetiX is a very fast and efficient independent research software that helps molecular imaging users easily and quickly produce 3D PET parametric images from any reconstructed 4D-PET data acquired on standard or large PET systems.

8.9 From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows

Participants: Thomas Gallouët, Andrea Natale, Gabriele Todeschi.

In [14] we introduce a time discretization for Wasserstein gradient flows based on the classical Backward Differentiation Formula of order two. The main building block of the scheme is the notion of geodesic extrapolation in the Wasserstein space, which in general is not uniquely defined. We propose several possible definitions for such an operation, and we prove convergence of the resulting scheme to the limit PDE, in the case of the Fokker-Planck equation. For a specific choice of extrapolation we also prove a more general result, that is convergence towards EVI flows. Finally, we propose a variational finite volume discretization of the scheme which numerically achieves second order accuracy in both space and time.

8.10 Metric extrapolation in the Wasserstein space

Participants: Thomas Gallouët, Andrea Natale, Gabriele Todeschi.

In [24] we study a variational problem providing a way to extend for all times minimizing geodesics connecting two given probability measures, in the Wasserstein space. This is simply obtained by allowing for negative coefficients in the classical variational characterization of Wasserstein barycenters. We show that this problem admits two equivalent convex formulations: the first can be seen as a particular instance of Toland duality and the second is a barycentric optimal transport problem. We propose an efficient numerical scheme to solve this latter formulation based on entropic regularization and a variant of Sinkhorn algorithm.

8.11 Regularity theory and geometry of unbalanced optimal transport

Participants: Thomas Gallouët, Roberta Ghezzi, Francois-Xavier Vialard.

In [23] using the dual formulation only, we show that the regularity of unbalanced optimal transport also called entropy-transport inherits from the regularity of standard optimal transport. We provide detailed examples of Riemannian manifolds and costs for which unbalanced optimal transport is this [http URL](#) all entropy-transport formulations, Wasserstein-Fisher-Rao (WFR) metric, also called Hellinger-Kantorovich, stands out since it admits a dynamic formulation, which extends the Benamou-Brenier formulation of optimal transport. After demonstrating the equivalence between dynamic and static formulations on a closed Riemannian manifold, we prove a polar factorization theorem, similar to the one due to Brenier and Mc-Cann. As a byproduct, we formulate the Monge-Ampère equation associated with WFR metric, which also holds for more general costs. Last, we study the link between c-convex functions for the cost induced by the WFR metric and the cost on the cone. The main result is that the

weak Ma-Trudinger-Wang condition on the cone implies the same condition on the manifold for the cost induced by WFR.

8.12 Monge Ampère gravity: from the large deviation principle to cosmological simulations through optimal transport

Participants: Bruno Levy, Yann Brenier, Roya Mohayaee.

In [15] we study Monge-Ampère gravity (MAG) as an effective theory of cosmological structure formation through optimal transport theory. MAG is based on the Monge-Ampère equation, a nonlinear version of the Poisson equation, that relates the Hessian determinant of the potential to the density field. We explain how MAG emerges from a conditioned system of independent and indistinguishable Brownian particles, through the large deviation principle, in the continuum limit. To numerically explore this highly non-linear theory, we develop a novel N-body simulation method based on semi-discrete optimal transport. Our results obtained from the very first N-body simulation of Monge-Ampère gravity with over 100 millions particles show that on large scales, Monge-Ampère gravity is similar to the Newtonian gravity but favours the formation of anisotropic structures such as filaments. At small scales, MAG has a weaker clustering and is screened in high-density regions. Although here we study the Monge-Ampère gravity as an effective rather than a fundamental theory, our novel highly-performant optimal transport algorithm can be used to run high-resolution simulations of a large class of modified theories of gravity, such as Galileons, in which the equations of motion are second-order and of Monge-Ampère type.

8.13 Faster is Slower effect for evacuation processes: a granular standpoint

Participants: Sylvain Faure, Bertrand Maury.

In [9] we studied the so-called Faster is Slower (FIS) effect which is observed in some particular real-life or experimental situations. In the context of an evacuation process, it expresses that increasing the speed (or, more generally, the competitive-ness) of individuals may induce a reduction of the flow through the exit door. We propose here a parameter-free model to reproduce and investigate this effect (more precisely its backward “Slower is Faster” equivalent). In spite of its non-smooth character, which makes it difficult to analyze, this granular approach is based on very basic ingredients in terms of behavior. In its native, purely asocial version, individuals are represented by hard-discs, each of which has a desired velocity, and the actual velocity is built as the projection of this field on the set of admissible velocities (which respect the non-overlapping constraints). We implement the slower effect by introducing here an extra step to account for the fact that individuals refrain from pushing, and therefore tend to reduce their desired velocity accounting for the velocities of people upfront. The present paper has two objectives: establish the relevance of this model by showing that it satisfactorily reproduces various empirical effects in highly crowded evacuations with various levels of competitiveness, and explore how it can be implemented to recover and explain the FIS effect. In this spirit, we confront this Inhibition-Based (IB) model to experimental data, focusing on the Faster is Slower effect. We show in particular that this approach makes it possible to accurately recover the effect of competitiveness upon power-law distributions of time lapses which have been experimentally observed. We also study the effect of mixed behaviors, by introducing a two-population model using both approaches. We investigate in particular the effect upon evacuation efficiency of the ratio between competitive agents and non-competitive ones. In a similar context, we investigate the role of an obstacle placed upstream the exit upon evacuation efficiency.

9 Partnerships and cooperations

9.1 International initiatives

Karma

Title: Optimal Transport geometry and asymptotics

Duration: 2024 -> 2027

Coordinators: Thomas Gallouët and Brendan Pass (pass@ualberta.ca)

Partners:

- University of British Columbia Vancouver (Canada)

Inria contact: Thomas Gallouët

Summary: This research project aims to join the forces of two major teams working on Optimal transport worldwide. Sharing our experiences we will tackle several interesting and challenging questions related to Optimal transport. It is composed of three main axes. The first direction is to better understand the geometry inherits from the ground cost function. The second direction focuses on the very dynamic research field concerning the asymptotic of Entropy OT when the regularization parameter goes to zero. The third direction is the study of extension of Optimal Transport such as Multi-marginal Optimal Transport Problems, an important subject for theParMA, as well as the weak Optimal transport in which falls for exemple the martingale Optimal Transport a domain of expertise of the Canadians partners.

Participants within the Inria ParMA team:

Participants: Thomas Gallouët, Luca Nenna, Yann Brenier, Quentin Mérigot, Katharina Eichinger, Adrien Cancès, Louis Tocquec, Benjamin Capdeville.

9.2 International research visitors

9.2.1 Visits of international scientists

Other international visits to the team

David Bourne

Status professor

Institution of origin: Heriot-Watt University

Country: UK

Dates: 27/May/2024 01/June/2024

Context of the visit: Collaboration with Q. Mérigot and T. Gallouët on reconstructing Laguerre diagrams from partial data

Mobility program/type of mobility: research stay

Théo Lavier**Status** PhD**Institution of origin:** Heriot-Watt University**Country:** UK**Dates:** 17/June/2024 06/July/2024**Context of the visit:** Collaboration with Q. Mériqot on adapting semi-discrete optimal transport solvers to compressible semi-geostrophic equations**Mobility program/type of mobility:** research stay**9.3 National initiatives**

- Quentin Mériqot and Thomas Gallouët are members of the PEPR IA-EDP led by Antonin Chambolle (Inria Mokaplan). Quentin Mériqot is co-PI.
- Luca Nenna is PI of the ANR GOTA.
- Bruno Levy is the PI of the Inria AEX GEOMERIX.
- Quentin Mériqot has an IUF.

9.4 Regional initiatives

Luca Nenna is PI of one PGMO and one HCODE project. Quentin Meriqot and Thomas Gallouët are members of the respectively PGMO/HCODE project.

10 Dissemination**10.1 Promoting scientific activities****10.1.1 Scientific events: organisation****Member of the organizing committees**

Thomas Gallouët

- co-organized the workshop [Finite Volume and Optimal Transport: past, present and perspectives](#) in November 2024.
- organized the Kick-off meeting of the Inria ParMA team.

Luca Nenna

- co-organized an Optimal Transport session at PGMO days, November 2024.
- co-organized [Moka10](#), June 2024.
- co-organized an Optimal Transport session at CANUM, May 2024.

Quentin Mériqot

- Organized a Workshop in Paris, on Particle Systems in Dynamics, Optimization, and Learning (march 2024)
- Organized the annual meeting of the ANEDP team of Laboratoire de mathématiques d'Orsay.

10.1.2 Journal

Member of the editorial boards

Quentin Mérigot is editor in the following journals:

- ESAIM: M2AN (since 2023),
- SIAM Journal on Imaging Science (since 2022)
- SMF Panoramas et Synthèses (since 2021),
- ESAIM: ProcS (Editor in Chief, since 2021)

Reviewer - reviewing activities

In 2024 Thomas Gallouët reviewed for the following journals : CPDE, Annales Henri Lebesgue, M2AN.

In 2024 Luca Nenna reviewed for the following journals : MOR, SIMA, JAMS, JFA.

In 2024, Quentin Mérigot reviewed for the following journals: Annales scientifiques de l'ENS, IMRN, Journal de l'école Polytechnique, Journal de l'AMS, JFA

10.1.3 Invited talks

In 2024 Thomas Gallouët gave the following presentations:

- Dec 2024. Collège de France, Séminaire de Mathématiques Appliquées, Paris. Séminaire de la chaire de Pierre Louis Lions. Enregistrement disponible [ici](#).
- June 2024. **Conference MOKA10**, Fontainebleau, France, Invited speaker.
- March 2024. **Workshop au centre Lagrange**, Paris, *Particle Systems in Dynamics, Optimization, and Learning*, Invited speaker.

In 2024 Quentin Merigot gave the following presentations:

- Jan 2024: Applications of Optimal Transportation, Oberwolfach (janvier 2024),
- April 2024: Optimal transport: theory and applications, Cargèse (avril 2024),
- June 2024: **Conference MOKA10**
- June 2024: BIRS Granada – PDE Methods in machine learning,
- November 2024: Conference on optimal transport and finite volumes Orsay
- Décembre 2024: Optimal transport and applications, Pisa

In 2024 Bertrand Maury gave the following presentations:

- Mars 2024: Particles2024, Institut Lagrange
- June 2024: Collective Motion, Cargèse,
- Mars 2024: Espaces mathématiques, espaces géographiques, CNAM Paris,
- Octobre 2024, métrique de Wasserstein, Collège de France
- Mars 2024, flots de gradient Wasserstein en physique, RNL 2024
- avril 2024, Transport Optimal pour les équations d'Euler sans pression, Journées EDP de Créteil

In 2024 Luca Nenna gave the following presentations:

- August 2024, Kantorovich initiative meeting, UBC, Vancouver.
- July 2024 International Conference in Basic Science, Benjing.
- June 2024, 4th Italian Meeting on Probability and Mathematical Statistics, Rome.
- May 2024, CANUM, Île de Re.
- February 2024, EMC2 seminar, Sorbonne university.
- January 2024, Kick-off meeting ANR SOCOT.
- January 2024, ANEDP seminar, Université de Nice.
- January 2024, Numerical methods for optimal transport problems, mean field games and multi-agent dynamics, Universidad Federico Santa María, Valparaiso, Chile.

In 2024 Bruno Levy gave the following presentations:

- 02/01/2024 [Journées du pole calcul du LIS](#)
- 02/20/2024 [IRMA PDE Seminar/Strasbourg](#)
- 05/15/2024 [Shapes seminar, Jussieu](#)
- 08/11-16/2024 [BIRS workshop Optimal Transport and Dynamics \(Oaxaca\)](#)
- 09/17-20/2024 [RING meeting, Nancy](#)
- 11/07/2024 Rencontres Recherche a risque: enjeux et perspectives au ministère de la recherche et de l'enseignement supérieur, présentation du programme Inria Quadrant
- 11/07-08/2024 [First annual soft Risc-V systems workshop](#), presenting FemtoRV, an educational, easy to understand Risc-V softcore, presenting [FemtoRV](#), an educational, easy to understand Risc-V softcore
- 11/18/2024: Institute of Applied Mathematics and Information Technologies (IMATI, Italy), giving an invited talk: High-risk high-gain research: the challenge of inter-disciplinarity
- 11/29/2024: Monge-Ampère gravity and galileons day at Institut d'Astrophysique de Paris, gave two talks: Adjoint Newton Raphson optimal transport for red-shift distortion and Towards giga-scale semi-discrete optimal transport
- [GDR IASIS, Optimal Transport Days](#), keynote talk on semi-discrete optimal transport with 10^9 points and beyond, how and why

In 2024 Yann Brenier gave the following colloquiums:

- 30/02/25 Padoue, FROM FLUIDS TO COMBINATORICS AND VICE VERSA VIA OPTIMAL TRANSPORT, Université de Padoue.
- 24/03/2024 BOCCONI, Milan, GRAVITATION AND OPTIMAL TRANSPORT, Université.

In 2024 Yann Brenier gave the following presentations at conferences:

- 2-6/12/24, OPTIMAL TRANSPORTATION AND APPLICATIONS, CENTRO DE GIORGI, PISE, GRAVITATION: FROM EINSTEIN TO MONGE-AMPERE...THROUGH EULER
- 23-27/09/24, material sciences, charles University, Prague, « INITIAL VALUE PROBLEMS BY SPACE-TIME CONVEX OPTIMIZATION »
- 8-11/07/24, MODERN PERSPECTIVES IN APPLIED MATHEMATICS: THEORY AND NUMERICS OF PDEs, ETH ZURICH. « INITIAL VALUE PROBLEMS BY SPACE-TIME CONVEX OPTIMIZATION »

- 4-6/06/24, MOKA10-JOURNEES MOKAPLAN, Ecoles des Mines, Fontainebleau. « SOLVING IVP BY CONVEX MINIMIZATION »
- 25-29/03/24, CONFERENCE SMAI-MODE, ENS-LYON , Mini-cours. « HYDRODYNAMIQUE ET TRANSPORT OPTIMAL »
- 04-09/02/24, APPLICATIONS OF OPTIMAL TRANSPORT , MFO, Oberwolfach. « Optimal Transport à la Benamou-Brenier for the Gravitational Vlasov-Poisson Initial Value Problem »
- 16-19/01/24, KINETIC AND HYDRODYNAMIC PDEs, Conference in honour of François Golse's 60th birthday, ETH, ZURICH.

10.1.4 Scientific expertise

Luca Nenna reviewed a grant proposal for DFG (German Research Foundation) and Inria Quadrant program.

10.1.5 Research administration

Luca Nenna is the PI. of the ANR GOTA.

Quentin Mérigot was head of the ANEDP team in Laboratoire de mathématiques d'Orsay, and the co-PI of PEPR EDP-AI (project led by Antonin Chambolle) and Inria startup studio project RAYMAPR (led by Boris Thibert)

Bruno Levy is the scientific director of national [Programme Inria Quadrant](#) for High-Risk High-Reward research

10.2 Teaching - Supervision - Juries

10.2.1 Teaching

Thomas Gallouët taught 48h at Paris-Saclay university, including a course on Optimal Transport in M2.

Luca Nenna taught 140h at Paris-Saclay university, including two courses in M2: Calculus of variation and Optimization.

Quentin Mérigot taught 96h at ENS.

10.2.2 Supervision

Thomas Gallouët supervised the following Phd Student

- Erwan Stämpfli (third year, co-direction with Y. Brenier)
- Siwan Boufadene (second year, co-direction with F.X. Vialard)
- Benjamin Capdeville (first year, co-direction with L. Monsaingeon)
- Quentin Giton (first year, co-direction with Z. Kobeissi)

Luca Nenna supervised the following Phd Student

- Adrien Cances (second year, co-direction with Q. Mérigot)
- Louis Tocquec (first) year, co-direction with P. Pegon)
- Elise Bonnet-Weill (first year, co-direction with V. Ehrlacher)

Quentin Mérigot supervised the following PhD students:

- Adrien Cances (second year, co-direction with L. Nenna)
- Christophe Vauthier (first year, co-direction with Anna Korba)

10.2.3 Juries

Quentin Mérigot participated to the following juries:

- Jean-Baptiste Volatier (ONERA, 2024, PhD, president)
- Laurent Pfeiffer (CentraleSupélec, 2025, HDR, reviewer)

Thomas Gallouët participated to the following jury:

- Mustapha Bahari (Université Mohammed VI Polytechnique et Université côte d'Azur, 2024, PHd, reviewer)

Sylvain Faure participated to the following jury:

- Adel Taleb (EPHE, Université PSL, 2024, PHd)

Bruno Levy participated to the following jury:

- Charles Dapogny (Habilitation Thesis)

Yann Brenier participated to the following juries

- Luca Talamini (directeur Fabio Ancona), Université de Padoue, 28/02/25, Rapporteur.
- Jules Pitcho (directeur Nikolay Tzvetkov), ENS Lyon, 22/20/24, Présient du jury.

10.3 Popularization

10.3.1 Productions (articles, videos, podcasts, serious games, ...)

Bruno Levy wrote the following articles:

- 03/22/2024: Blog Binaire-Le Monde: [S'il te plait, dessine moi la puce du futur](#)
- 07/28/2024: [Emmy Noether et ses fabuleux théorèmes](#) (the conversation)
- 10/24/2024: participation à la rédaction de [A la recherche du transport optimal](#), sur l'équipe ParMA (site web Inria)

10.3.2 Participation in Live events

Bruno Levy participated to the following events:

- 01/19/2024: [Nuits de la lecture, si on lisait des mathématiques](#), organisée par [Les Maths en Scène](#)
- 04/12/2024: Printemps des mathématiques, Lycée Rosa Parks, Thionville ([livret d'activités](#))
- 10/4-6/2024: animation à la citée des sciences et de l'industrie sur [les mystères de l'Univers](#)

11 Scientific production

11.1 Major publications

- [1] F. Al Reda, S. Faure, B. A. Maury and E. Pinsard. ‘Faster is Slower effect for evacuation processes: a granular standpoint’. In: *Journal of Computational Physics*. VSI: In honor of Roland Glowinski 504 (2024), p. 112861. DOI: [10.1016/j.jcp.2024.112861](https://doi.org/10.1016/j.jcp.2024.112861). URL: <https://hal.science/hal-04490266>.
- [2] F. Besson and S. Faure. ‘PET KinetiX—A Software Solution for PET Parametric Imaging at the Whole Field of View Level’. In: *Journal of Imaging Informatics in Medicine* 37.2 (10th Jan. 2024), pp. 842–850. DOI: [10.1007/s10278-023-00965-z](https://doi.org/10.1007/s10278-023-00965-z). URL: <https://hal.science/hal-04946630>.
- [3] S. Di Marino, M. Lewin and L. Nenna. ‘Grand-Canonical Optimal Transport’. In: *Archive for Rational Mechanics and Analysis* 249 (2025), art. 12. DOI: [10.1007/s00205-024-02080-x](https://doi.org/10.1007/s00205-024-02080-x). URL: <https://hal.science/hal-03536401>.
- [4] S. Di Marino, M. Lewin and L. Nenna. ‘Ground state energy is not always convex in the number of electrons’. In: *Journal of Physical Chemistry A* 128.49 (Nov. 2024), pp. 10697–10706. DOI: [10.1021/acs.jpca.4c06345](https://doi.org/10.1021/acs.jpca.4c06345). URL: <https://hal.science/hal-04698997> (cit. on p. 14).
- [5] T. Gallouët, A. Natale and G. Todeschi. ‘From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows’. In: *Mathematics of Computation* 93 (2024), pp. 2769–2810. DOI: [10.1090/mcom/3951](https://doi.org/10.1090/mcom/3951). URL: <https://hal.science/hal-03790981>.
- [6] C. Letrouit and Q. Mérigot. *Gluing methods for quantitative stability of optimal transport maps*. 5th Jan. 2025. URL: <https://hal.science/hal-04864913>.
- [7] B. Lévy, Y. Brenier and R. Mohayaee. ‘Monge Ampère gravity: from the large deviation principle to cosmological simulations through optimal transport’. In: *Physical Review D* 110.6 (2024), p. 063550. DOI: [10.1103/PhysRevD.110.063550](https://doi.org/10.1103/PhysRevD.110.063550). URL: <https://hal.science/hal-04590343>.
- [8] L. Nenna and P. Pegon. ‘Convergence rate of entropy-regularized multi-marginal optimal transport costs’. In: *Canadian Journal of Mathematics = Journal Canadien de Mathématiques* (15th Mar. 2024). DOI: [10.4153/S0008414X24000257](https://doi.org/10.4153/S0008414X24000257). URL: <https://hal.science/hal-04154453> (cit. on p. 14).

11.2 Publications of the year

International journals

- [9] F. Al Reda, S. Faure, B. A. Maury and E. Pinsard. ‘Faster is Slower effect for evacuation processes: a granular standpoint’. In: *Journal of Computational Physics*. VSI: In honor of Roland Glowinski 504 (2024), p. 112861. DOI: [10.1016/j.jcp.2024.112861](https://doi.org/10.1016/j.jcp.2024.112861). URL: <https://hal.science/hal-04490266> (cit. on pp. 16, 18).
- [10] F. Besson and S. Faure. ‘PET KinetiX—A Software Solution for PET Parametric Imaging at the Whole Field of View Level’. In: *Journal of Imaging Informatics in Medicine* 37.2 (10th Jan. 2024), pp. 842–850. DOI: [10.1007/s10278-023-00965-z](https://doi.org/10.1007/s10278-023-00965-z). URL: <https://hal.science/hal-04946630> (cit. on pp. 13, 16).
- [11] S. Di Marino, M. Lewin and L. Nenna. ‘Grand-Canonical Optimal Transport’. In: *Archive for Rational Mechanics and Analysis* 249 (2025), art. 12. DOI: [10.1007/s00205-024-02080-x](https://doi.org/10.1007/s00205-024-02080-x). URL: <https://hal.science/hal-03536401>.
- [12] S. Di Marino, M. Lewin and L. Nenna. ‘Ground state energy is not always convex in the number of electrons’. In: *Journal of Physical Chemistry A* 128.49 (Nov. 2024), pp. 10697–10706. DOI: [10.1021/acs.jpca.4c06345](https://doi.org/10.1021/acs.jpca.4c06345). URL: <https://hal.science/hal-04698997>.
- [13] H. Ennaji, Q. Mérigot, L. Nenna and B. Pass. ‘Robust risk management via multi-marginal optimal transport’. In: *Journal of Optimization Theory and Applications* 202.2 (9th May 2024), pp. 554–581. DOI: [10.1007/s10957-024-02438-x](https://doi.org/10.1007/s10957-024-02438-x). URL: <https://hal.science/hal-03852245> (cit. on p. 15).

- [14] T. Gallouët, A. Natale and G. Todeschi. ‘From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows’. In: *Mathematics of Computation* 93 (2024), pp. 2769–2810. DOI: [10.1090/mcom/3951](https://doi.org/10.1090/mcom/3951). URL: <https://hal.science/hal-03790981> (cit. on p. 17).
- [15] B. Lévy, Y. Brenier and R. Mohayaee. ‘Monge Ampère gravity: from the large deviation principle to cosmological simulations through optimal transport’. In: *Physical Review D* 110.6 (2024), p. 063550. DOI: [10.1103/PhysRevD.110.063550](https://doi.org/10.1103/PhysRevD.110.063550). URL: <https://hal.science/hal-04590343> (cit. on pp. 11, 18).
- [16] H. A. Martin, A. Mangeney, X. Jia, B. A. Maury, A. Lefebvre-Lepot, Y. Maday and P. Dérand. ‘Ultrasound-induced dense granular flows: a two-time scale modeling’. In: *Journal of Fluid Mechanics* 1004 (30th Jan. 2025), A10. DOI: [10.1017/jfm.2024.1159](https://doi.org/10.1017/jfm.2024.1159). URL: <https://hal.science/hal-04454075>.
- [17] L. Nenna and P. Pegon. ‘Convergence rate of entropy-regularized multi-marginal optimal transport costs’. In: *Canadian Journal of Mathematics = Journal Canadien de Mathématiques* (15th Mar. 2024). DOI: [10.4153/S0008414X24000257](https://doi.org/10.4153/S0008414X24000257). URL: <https://hal.science/hal-04154453>.

Edition (books, proceedings, special issue of a journal)

- [18] *Editorial of CEMRACS 2022* 77 (18th Nov. 2024). DOI: [10.1051/proc/202477001](https://doi.org/10.1051/proc/202477001). URL: <https://hal.science/hal-04936536>.

Doctoral dissertations and habilitation theses

- [19] L. Nenna. ‘On some generalisations of Optimal Transport problem’. Paris Saclay University, 6th Mar. 2024. URL: <https://inria.hal.science/tel-04554531>.

Reports & preprints

- [20] J.-B. Casteras, L. Monsaingeon and L. Nenna. *Large deviations for sticky-reflecting Brownian motion with boundary diffusion*. 18th Jan. 2025. URL: <https://inria.hal.science/hal-04895784>.
- [21] C. Dapogny, B. Lévy and E. Oudet. *A Lagrangian shape and topology optimization framework based on semi-discrete optimal transport*. 23rd Sept. 2024. URL: <https://hal.science/hal-04705538>.
- [22] V. Ehrlicher and L. Nenna. *A sparse approximation of the Lieb functional with moment constraints*. 17th June 2024. URL: <https://hal.science/hal-04131066>.
- [23] T. Gallouët, R. Ghezzi and F.-X. Vialard. *Regularity theory and geometry of unbalanced optimal transport*. 27th June 2024. URL: <https://hal.science/hal-03498098> (cit. on p. 17).
- [24] T. O. Gallouët, A. Natale and G. Todeschi. *Metric extrapolation in the Wasserstein space*. 19th July 2024. URL: <https://hal.science/hal-04647125> (cit. on p. 17).
- [25] J. Hiew, L. Nenna and B. Pass. *An ordinary differential equation for entropic optimal transport and its linearly constrained variants*. 28th Mar. 2024. URL: <https://inria.hal.science/hal-04525602> (cit. on p. 15).
- [26] C. Letrouit and Q. Mérigot. *Gluing methods for quantitative stability of optimal transport maps*. 5th Jan. 2025. URL: <https://hal.science/hal-04864913> (cit. on p. 15).
- [27] L. de Pascale and L. Nenna. *A variational formulation of a Multi-Population Mean Field Games with non-local interactions*. 6th Aug. 2024. URL: <https://inria.hal.science/hal-04668323> (cit. on p. 14).
- [28] C. Vauthier, Q. Merigot and A. Korba. *Properties of Wasserstein Gradient Flows for the Sliced-Wasserstein Distance*. 10th Feb. 2025. URL: <https://hal.science/hal-04963968>.

Software

- [29] [SW] H. Bloch, S. Faure, L. Gouarin, Q. Houssier and A. Lefebvre-Lepot, *SCoPI* 21st May 2024. LIC: BSD 3-Clause "New" or "Revised" License. HAL: ([hal-04582379](https://hal.science/hal-04582379)), URL: <https://hal.science/hal-04582379>, VCS: <https://github.com/hpc-maths/scopi>, SWHID: ([swh:1:dir:e9a205ba0833cc254930748cb01f323bd004bc28](https://sw.hydro.ac.uk/swh:1:dir:e9a205ba0833cc254930748cb01f323bd004bc28); origin=<https://github.com/hpc-maths/scopi>; visit=[swh:1:snp:492c6274ba96d3dbc36382a7da41070fcf1e2c4e](https://sw.hydro.ac.uk/swh:1:snp:492c6274ba96d3dbc36382a7da41070fcf1e2c4e); anchor=[swh:1:rev:6b885e3f0a43abda55cae5c52b2e1e7e1275abc1](https://sw.hydro.ac.uk/swh:1:rev:6b885e3f0a43abda55cae5c52b2e1e7e1275abc1)).

11.3 Cited publications

- [30] M. Agueh and G. Carlier. ‘Barycenters in the Wasserstein space’. In: *SIAM Journal on Mathematical Analysis* 43.2 (2011), pp. 904–924 (cit. on p. 9).
- [31] A. Alfonsi, R. Coyaud and V. Ehrlacher. ‘Constrained overdamped Langevin dynamics for symmetric multimarginal optimal transportation’. In: *Mathematical Models and Methods in Applied Sciences* 32.03 (2022), pp. 403–455 (cit. on p. 10).
- [32] A. Alfonsi, R. Coyaud, V. Ehrlacher and D. Lombardi. ‘Approximation of optimal transport problems with marginal moments constraints’. In: *Mathematics of Computation* 90.328 (2021), pp. 689–737 (cit. on p. 10).
- [33] L. Ambrosio, N. Gigli and G. Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2005 (cit. on p. 7).
- [34] J.-D. Benamou and Y. Brenier. ‘A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem’. In: *Numerische Mathematik* 84.3 (2000), pp. 375–393 (cit. on p. 4).
- [35] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna and G. Peyré. ‘Iterative Bregman projections for regularized transportation problems’. In: *SIAM Journal on Scientific Computing* 37.2 (2015), A1111–A1138 (cit. on p. 10).
- [36] J.-D. Benamou, G. Carlier and L. Nenna. ‘A numerical method to solve multi-marginal optimal transport problems with Coulomb cost’. In: *Splitting Methods in Communication, Imaging, Science, and Engineering*. Springer, 2016, pp. 577–601 (cit. on p. 10).
- [37] J.-D. Benamou, G. Carlier and L. Nenna. ‘Generalized incompressible flows, multi-marginal transport and Sinkhorn algorithm’. In: *Numerische Mathematik* 142.1 (2019), pp. 33–54 (cit. on p. 10).
- [38] J.-D. Benamou, T. Gallouët and F.-X. Vialard. ‘Second order models for optimal transport and cubic splines on the Wasserstein space’. In: *Foundations of Computational Mathematics* (Oct. 2019). DOI: [10.1007/s10208-019-09425-z](https://doi.org/10.1007/s10208-019-09425-z). URL: <https://hal.archives-ouvertes.fr/hal-01682107> (cit. on p. 8).
- [39] Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee and A. Sobolevski. ‘Reconstruction of the early Universe as a convex optimization problem’. In: *Monthly Notices of the Royal Astronomical Society* 346 (Dec. 2003), pp. 501–524. DOI: [10.1046/j.1365-2966.2003.07106.x](https://doi.org/10.1046/j.1365-2966.2003.07106.x) (cit. on p. 8).
- [40] Y. Brenier. ‘Derivation of the Euler Equations from a Caricature of Coulomb Interaction’. In: *Communications in Mathematical Physics* 212.1 (2000), pp. 93–104 (cit. on p. 3).
- [41] Y. Brenier. ‘The least action principle and the related concept of generalized flows for incompressible perfect fluids’. In: *Journal of the American Mathematical Society* 2.2 (1989), pp. 225–255 (cit. on p. 9).
- [42] G. Buttazzo, L. De Pascale and P. Gori-Giorgi. ‘Optimal-transport formulation of electronic density-functional theory’. In: *Physical Review A* 85.6 (2012), p. 062502 (cit. on p. 9).
- [43] G. Carlier and I. Ekeland. ‘Matching for teams’. In: *Economic theory* 42.2 (2010), pp. 397–418 (cit. on p. 9).
- [44] P.-A. Chiappori, R. J. McCann and B. Pass. ‘Multi-to One-Dimensional Optimal Transport’. In: *Communications on Pure and Applied Mathematics* 70.12 (2017), pp. 2405–2444 (cit. on p. 5).

- [45] C. Cotar, G. Friesecke and C. Klüppelberg. ‘Density functional theory and optimal transportation with Coulomb cost’. In: *Communications on Pure and Applied Mathematics* 66.4 (2013), pp. 548–599 (cit. on p. 9).
- [46] M. J. Cullen and R. J. Purser. ‘An extended Lagrangian theory of semi-geostrophic frontogenesis’. In: *Journal of Atmospheric Sciences* 41.9 (1984), pp. 1477–1497 (cit. on p. 4).
- [47] M. Cuturi. ‘Sinkhorn distances: Lightspeed computation of optimal transport’. In: *Advances in neural information processing systems* 26 (2013) (cit. on p. 4).
- [48] S. Di Marino, M. Lewin and L. Nenna. *Grand-canonical optimal transport*. 2022. arXiv: 2201.06859 (cit. on p. 10).
- [49] A. Galichon and B. Salanie. ‘Matching with Trade-offs: Preferences over Competing Characteristics’. In: (2010) (cit. on p. 4).
- [50] T. Gallouët, R. Ghezzi and F.-X. Vialard. *Regularity theory and geometry of unbalanced optimal transport*. 2021. DOI: 10.48550/ARXIV.2112.11056. URL: <https://arxiv.org/abs/2112.11056> (cit. on p. 4).
- [51] T. O. Gallouët, Q. Merigot and A. Natale. ‘Convergence of a Lagrangian discretization for barotropic fluids and porous media flow’. In: *SIAM Journal on Mathematical Analysis* 54.3 (2022), pp. 2990–3018 (cit. on p. 7).
- [52] T. O. Gallouët and Q. Mérigot. ‘A Lagrangian scheme à la Brenier for the incompressible Euler equations’. In: *Foundations of Computational Mathematics* 18.4 (2018), pp. 835–865 (cit. on p. 7).
- [53] B. Iooss and P. Lemaître. ‘A review on global sensitivity analysis methods’. In: *Uncertainty management in simulation-optimization of complex systems*. Springer, 2015, pp. 101–122 (cit. on pp. 9, 10).
- [54] R. Jordan, D. Kinderlehrer and F. Otto. ‘The variational formulation of the Fokker–Planck equation’. In: *SIAM journal on mathematical analysis* 29.1 (1998), pp. 1–17 (cit. on p. 7).
- [55] S. Kondratyev, L. Monsaingeon and D. Vorotnikov. ‘A new optimal transport distance on the space of finite Radon measures’. In: *arXiv preprint arXiv:1505.07746* (2015) (cit. on p. 4).
- [56] B. Maury and A. Preux. ‘Pressureless Euler equations with maximal density constraint: a time-splitting scheme’. In: *Topological Optimization and Optimal Transport: In the Applied Sciences* 17 (2017), p. 333 (cit. on p. 7).
- [57] B. Maury, A. Roudneff-Chupin, F. Santambrogio and J. Venel. ‘Handling congestion in crowd motion modeling’. In: *arXiv preprint arXiv:1101.4102* (2011) (cit. on p. 7).
- [58] R. J. McCann and B. Pass. ‘Optimal transportation between unequal dimensions’. In: *Archive for Rational Mechanics and Analysis* 238.3 (2020), pp. 1475–1520 (cit. on p. 5).
- [59] Q. Mérigot. ‘A multiscale approach to optimal transport’. In: *Computer Graphics Forum*. Vol. 30. 5. Wiley Online Library, 2011, pp. 1583–1592 (cit. on p. 4).
- [60] Q. Mérigot, A. Delalande and F. Chazal. ‘Quantitative stability of optimal transport maps and linearization of the 2-Wasserstein space’. In: *International Conference on Artificial Intelligence and Statistics*. PMLR, 2020, pp. 3186–3196 (cit. on pp. 4, 5, 9).
- [61] Q. Mérigot and J.-M. Mirebeau. ‘Minimal geodesics along volume preserving maps, through semi-discrete optimal transport’. In: *SIAM Journal on Numerical Analysis* 54.6 (Nov. 2016), pp. 3465–3492. DOI: 10.1137/15M1017235. URL: <https://hal.archives-ouvertes.fr/hal-01152168> (cit. on p. 8).
- [62] L. Métivier, R. Brossier, Q. Mérigot, E. Oudet and J. Virieux. ‘Measuring the misfit between seismograms using an optimal transport distance: Application to full waveform inversion’. In: *Geophysical Supplements to the Monthly Notices of the Royal Astronomical Society* 205.1 (2016), pp. 345–377 (cit. on p. 8).
- [63] L. Métivier, R. Brossier, Q. Mérigot, E. Oudet and J. Virieux. ‘Measuring the misfit between seismograms using an optimal transport distance: application to full waveform inversion’. In: *Geophysical Journal International* 205.1 (Feb. 2016), pp. 345–377. DOI: 10.1093/gji/ggw014. URL: <https://hal.archives-ouvertes.fr/hal-01499531> (cit. on p. 3).

-
- [64] L. Nenna and B. Pass. 'Variational problems involving unequal dimensional optimal transport'. In: *Journal de Mathématiques Pures et Appliquées* 139 (2020), pp. 83–108 (cit. on p. 5).
 - [65] E. Otto. *Double degenerate diffusion equations as steepest descent*. Citeseer, 1996 (cit. on p. 7).
 - [66] E. Otto. 'The geometry of dissipative evolution equations: the porous medium equation'. In: (2001) (cit. on p. 7).
 - [67] B. Pass. 'Uniqueness and Monge solutions in the multimarginal optimal transportation problem'. In: *SIAM Journal on Mathematical Analysis* 43.6 (2011), pp. 2758–2775 (cit. on p. 10).