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ACTIVITY REPORT

Project-Team

TROPICAL

Tropical methods: structures, algorithms and interactions

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP)

DOMAIN

Applied Mathematics, Computation and Simulation

THEME

Optimization and control of dynamic systems

Inria

Contents

Project-Team TROPICAL

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2 Overall objectives

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the Tropical project, namely, "structures, algorithms, and interactions", refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.1 Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [66], combinatorial optimization [75], or automata theory [113]. They also arose in mathematical physics and asymptotic analysis [104, 98]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [90, 129, 118, 96], in algebraic combinatorics [85], and in arithmetics [79]. Also, further applications of tropical methods have appeared, including optimal control [105], program invariant computation [58] and timed systems verification [102], and zero-sum games [1].

The term 'tropical' generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations ("log-limits sets" of Bergman, "Maslov dequantization", or "Viro deformation"). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [129, 109]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [124, 66, 93, 76].

Tropical methods are linked to the fields of positive systems and of metric geometry [111], [12]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert's projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [100], [2]. Such dynamical systems are of fundamental importance in the study of repeated games [108]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [80]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [6]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [125] or in a PDE setting [69]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [10], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [117, 122].

3 Research program

3.1 Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [82]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [95], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [54], [56], [2].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [83]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the "curse of dimensionality". A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called "max-plus basis methods" [84]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [106, 87]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, "highway hierarchies", etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [59] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [10] and discrete event systems [66, 78, 61]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [2] or metric geometry [11]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3 Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [64],[1], tropical semialgebraic sets [38], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [57], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

3.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \quad , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \quad , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [104], non-archimedean valuations, log-limit sets and Viro's patchworking method [129], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [88, 92].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [60],[7], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [38], [63], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4 Application domains

4.1 Discrete event systems (manufacturing systems, networks, emergency call centers)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [66]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [61].

4.2 Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3 Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [119]. Applications of tropical methods arise in particular from discrete optimization [76], [77], scheduling problems with and-or constraints [110], or product mix auctions [127].

4.4 Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [80], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called “templates” introduced by Sankaranarayanan, Sipma and Manna [121]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [58], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [86, 89] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [6]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5 Social and environmental responsibility

5.1 Impact of research results

The team has developed collaborations on the dimensioning of emergency call centers, with Préfecture de Police (Plate Forme d’Appels d’Urgence - PFAU - 17-18-112, operated jointly by Brigade de sapeurs pompiers de Paris and by Direction de la sécurité de proximité de l’agglomération parisienne) and also with the Emergency medical services of Assistance Publique – Hôpitaux de Paris (Centre 15 of SAMU75, 92, 93 and 94). This work is described further in Section 8.7.1. A recent extension of this work deals with the modelling of medical emergency services, with the project “URGE” which started at the fall 2022, in the framework of the joint INRIA & AP-HP “Bernoulli” lab.

6 Highlights of the year

- Plenary talk of S. Gaubert at the 35th International Workshop on Operator Theory and its Applications (IWOTA 2024), University of Kent, Canterbury, UK, on the topic of “Tropical geometry, operators and games”. The video is available here.

7 New software, platforms, open data

7.1 New software

7.1.1 Coq-Polyhedra

Name: Coq-Polyhedra

Keywords: Coq, Polyhedra, Automated theorem proving, Linear optimization

Scientific Description: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

Functional Description: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

URL: `https://github.com/nhojem/Coq-Polyhedra`

Publications: hal-01673390, hal-03151656, hal-03915661, hal-01967575, hal-01967576

Contact: Xavier Allamigeon

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Partners: CIFASIS, Ecole Polytechnique

7.1.2 EmergencyEval

Keywords: Dynamic Analysis, Simulation, Ocaml, Emergency, Firefighters, Police

Scientific Description: This software aims at enabling the definition of a Petri network execution semantic, as well as the instanciation and execution of said network using the aforedefined semantic.

The heart of the project dwells in its kernel which operates the step-by-step execution of the network, obeying rules provided by an oracle. This user-defined and separated oracle computes the information necessary to the kernel for building the next state using the current state. The base of our software is the framework for the instanciation and execution of Petri nets, without making assumptions regarding the semantic.

In the context of the study of the dynamics of emergency call centers, a second part of this software is the definition and implementation of the semantic of call centers modelized as Petri nets, and more specifically timed prioritized Petri nets. A module interoperating with the kernel enables to include all the operational specificities of call centers (urgency level, discriminating between operators and callers ...) while guaranteeing the genericity of the kernal which embeds the Petri net formalism as such.

Functional Description: In order to enable the quantitative study of the throughput of calls managed by emergency center calls and the assesment of various organisationnal configurations considered by the stakeholders (firefighters, police, medical emergency service of the 75, 92, 93 and 94 French departments), this software modelizes their behaviours by resorting to extensions of the Petri net formalism. Given a call transfer protocol in a call center, which corresponds to a topology and an execution semantic of a Petri net, the software generates a set of entering calls in accord with the empirically observed statistic ditributions (share of very urgent calls, conversation length), then simulates its management by the operators with respect to the defined protocol. Transitional regimes phenomenons (peak load, support) which are not yet handled by mathematical analysis could therefore be studied. The ouput of the software is a log file which is an execution trace of the simulation featuring extensive information in order to enable the analysis of the data for providing simulation-based insights for decision makers.

The software relies on a Petri net simulation kernel designed to be as modular and adaptable as possible, fit for simulating other Petri-net related phenomenons, even if their semantic differ greatly.

Contact: Xavier Allamigeon

Participants: Xavier Allamigeon, Benjamin Nguyen-Van-Yen

8 New results

8.1 Optimal control and zero-sum games

8.1.1 Tropical numerical methods for stochastic control problems

Participants: Marianne Akian.

We are interested here in the numerical solution of the dynamic programming equation of discrete time deterministic or stochastic control problems.

In several works in collaboration with Jean-Philippe Chancelier (CERMICS) and Benoit Tran, and included in the PhD thesis of Benoit Tran [126], we developed and studied algorithms combining the tropical or the max-plus based numerical method of McEneaney [107, 105], the stochastic max-plus scheme proposed by Zheng Qu [116], and the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [112]. In particular in [13], we considered a stochastic algorithm for deterministic control problems, and in [47], we considered stochastic control problems.

In [40], we also show that in the case of the dynamic programming equation associated to a partially observable Markov Decision Process (POMDP), the algorithm studied in [47] is similar to the so called point based algorithms developed in [114, 99, 123], which includes in particular SARSOP algorithm. In an ongoing work with Luz Pascal (PhD student at Queensland University of Technology, Australia), we are studying the convergence of SARSOP algorithm using the same techniques as in [47].

8.1.2 Highway hierarchies for Hamilton-Jacobi-Bellman (HJB) PDEs

Participants: Marianne Akian, Stéphane Gaubert.

Hamilton-Jacobi-Bellman equations arise as the dynamic programming equations of deterministic or stochastic optimal control problems. They allow to obtain the global optimum of these problems and to synthesize an optimal feedback control, leading to a solution robust against system perturbations. Several methods have been proposed in the literature to bypass the obstruction of curse of dimensionality of such equations, assuming a certain structure of the problem, and/or using "unstructured discretizations", that are not based on given grids. Among them, one may cite tropical numerical method, and probabilistic numerical method. On another direction, "highway hierarchies", developed by Sanders, Schultes and coworkers [81, 120], initially for applications to on-board GPS systems, are a computational method that allows one to accelerate Dijkstra algorithm for discrete time and state shortest path problems.

The aim of the PhD thesis of Shanqing Liu[101] (now in Brown University) was to develop new numerical methods to solve Hamilton-Jacobi-Bellman equations that are less sensitive to curse of dimensionality.

In [15], we have developed a multilevel fast-marching method, extending to the PDE case the idea of "highway hierarchies". Given the problem of finding an optimal trajectory between two given points, the method consists in refining the grid only in a neighborhood of the optimal trajectory, which is itself computed using an approximation of the value function on a coarse grid. The complexity of the method is analysed using a priori error bounds for the discretization of the Hamilton-Jacobi PDE. In [29], we obtain error estimates of order 1 for the semi-Lagrangian scheme of the Hamilton-Jacobi equation of the minimum time problem, under some regularity assumptions on the dynamics and the domain.

8.1.3 Escape Rate Games

Participants: Marianne Akian, Stéphane Gaubert, Loic Marchesini.

The aim of the PhD thesis of Loic Marchesini is to study a new class of repeated zero-sum games in which the payoff of one player is the escape rate of a dynamical system whose dynamics depends on the actions of both players. We suppose that the dynamics obtained from nonexpansive operators. Considering order preserving finite dimensional linear operators over the positive cone endowed with Hilbert's projective (semi-)metric, we recover the matrix multiplication games introduced by Asarin et al. [65], which generalize the joint spectral radius of sets of nonnegative matrices. See also [55]. We show in [28] that escape rate games have a value, which is characterized in terms of non-linear eigenproblems, providing a a two-player analogue of "Mañe's Lemma" in ergodic control.

8.1.4 Vanishing discount limit in optimal control

Participants: Stéphane Gaubert.

In a joint work with Piermarco Cannarsa, Cristian Mendico (Roma Tor Vergata), and Marc Quincampoix (Brest) [19], we characterize the vanishing discount limit of the value function of an optimal control problem, in the absence of controllability conditions. We show that the limit in the uniform topology, if it exists, is the maximal subsolution of a system of Hamilton-Jacobi equations. We also provide an analogue of this result for problems in discrete time.

8.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

8.2.1 Invariant Finsler metrics on symmetric spaces

Participants: Cormac Walsh.

This is joint work with Bas Lemmens (Kent).

We are interested in metrics on symmetric spaces, in particular Finsler metrics that are invariant under the symmetries of the space. In a previous paper [91], it was established that there is such a metric naturally associated to every representation of the symmetric space, which depends on the system of weights of the representation. More precisely, one takes the convex hull of the weights and interprets that as the dual ball in the Artin subspace. The lengths of all other vectors can then determined by invariance. We are investigating this correspondence between representations and metrics in more detail. We wish to answer the question, concretely, which known representations correspond to which invariant Finsler metrics?

We have also been looking more closely an especially interesting class of symmetric space: the bounded symmetric domains with their Carathéodory (or equivalently, Kobayashi) metric. Here, the metric restricts to the supremum norm on the Artin subspace. These symmetric spaces are particularly tractable due to their relation with JB*-triples—each one arises as the unit ball of such a triple. The algebraic structure of the triple turns out to be very helpful.

We have been studying how these spaces can be embedded in each other isometrically. We show [36] that a bounded symmetric domain cannot be isometrically embedded into one of lower rank. In the case where the rank of the target space is strictly larger, it turns out that there is a lot of flexibility in what the map can look like. When the two spaces have the same rank, however, the situation is much more rigid—we show that the map must respect any product structure of the domain, and furthermore, in the case where the domain is irreducible, that the map is either holomorphic or anti-holomorphic.

These results generalise work of Seo and Kim, who were working under the additional assumption that the maps in question are C^1-smooth. They made this assumption because they were using techniques from differential geometry, which require it. We are able to relax the assumption because, rather than the small scale geometry of the space, we consider the large scale geometry. In particular, a tool we use is the horofunction boundary—for bounded symmetric domains, this was determined in earlier work by Chu–Cueto-Avellaneda—Lemmens.

8.2.2 Isoperimety in Hilbert geometries

Participants: Amanda Bigel, Stéphane Gaubert, Constantin Vernicos, Cormac Walsh.

The PhD work of Amanda Bigel deals with the investigation of isoperimetric inequalities in the geometries of Funk and Hilbert's metric on convex domains.

8.3 Tropical algebra and convex geometry

8.3.1 Formalizing convex polyhedra in Coq

Participants: Xavier Allamigeon, Quentin Canu.

In a joint work with Pierre-Yves Strub (Meta), we have achieved the formal verification of a counter-example of Santos et al. to the so-called Hirsch Conjecture on the diameter of polytopes. In contrast with the pen-and-paper proof, our approach is entirely computational: we implement in Coq and prove correct an algorithm that explicitly computes, within the proof assistant, vertex-edge graphs of polytopes as well as their diameter. The originality of this certificate-based algorithm is to achieve a tradeoff between simplicity and efficiency. Simplicity is crucial in obtaining the proof of correctness of the algorithm. This proof splits into the correctness of an abstract algorithm stated over proof-oriented data types and the correspondence with a low-level implementation over computation-oriented data types. A special effort has been made to reduce the algorithm to a small sequence of elementary operations (e.g., matrix multiplications, basic routines on sets and graphs), in order to make the derivation of the correctness of the low-level implementation more transparent. Efficiency allows us to scale up to polytopes with a challenging combinatorics. For instance, we formally check the two counterexamples of Matschke, Santos and Weibel to the Hirsch conjecture, respectively 20- and 23-dimensional polytopes with 36 425 and 73 224 vertices involving rational coefficients with up to 40 digits in their numerator and denominator. We also illustrate the performance of the method by computing the list of vertices or the diameter of well-known classes of polytopes, such as (polars of) cyclic polytopes involved in McMullen's Upper Bound Theorem. This work has appeared in the proceedings of the conference CPP'23 [42].

Dealing with polyhedra and their faces raises the problem of formalizing order structures. We study this problem in a joint work [52] with Cyril Cohen (Inria), Kazuhiko Sakaguchi (Inria) and Pierre-Yves Strub (Meta). More precisely, using order structures in a proof assistant naturally raises the problem of working with multiple instances of a same structure over a common type of elements. This goes against the main design pattern of hierarchies used for instance in Coq's MathComp or Lean's mathlib libraries, where types are canonically associated to at most one instance and instances share a common overloaded syntax. We introduce new design patterns to leverage these issues, and apply them to the formalization of order structures in the MathComp library. A common idea in these patterns is underloading, i.e., a disambiguation of operators on a common type. In addition, our design patterns include a way to deal with duality in order structures in a convenient way. We hence formalize a large hierarchy which includes partial orders, semilattices, lattices as well as many variants. We finally pay a special attention to order substructures. We introduce a new kind of structure called prelattice. They are abstractions of semilattices, and allow us to deal with finite lattices and their sublattices within a common signature. As an application, we report on significant simplifications of the formalization of the face lattices of polyhedra in the Coq-Polyhedra library.

8.3.2 Linear algebra over systems

Participants: Marianne Akian, Stéphane Gaubert.

In a joint work with Louis Rowen (Univ. Bar Ilan), we study the properties of “systems”. The latter provide a general setting encompassing extensions of the tropical semifields and hyperfields. Moreover, they have the advantage to be well adapted to the study of linear or polynomial equations. In particular, in [14], we characterize the semiring systems which arise from hyperrings.

In [48], we are studying linear algebra properties over a generalization of “systems” called “T-pairs”. We are still working on improvements of the results of [48].

8.3.3 Roots over the symmetrized tropical semiring and eigenvalues of tropical symmetric matrices

Participants: Marianne Akian, Stéphane Gaubert.

The tropical semifield can be thought of as the image of a field with a non-archimedean valuation. It allows in this way to study the asymptotics of Puiseux series with complex coefficients. When dealing with Puiseux series with real coefficients and with its associated order, it is convenient to use the symmetrized tropical semiring introduced in [115] (see also [66]), and the signed valuation which associates to any series its valuation together with its sign.

In a work [49] which started during the postdoc of Hanieh Tavakolipour (Amirkabir University of Technology) in the team, we studied the roots's multiplicities and the factorization of polynomials over the symmetrized tropical semiring. We then deduced a Descartes' rule of sign over ordered valued fields. This builds in particular on [67] (for multiplicities) and on [38].

More recently, in a work with Dariush Kiani and Hanieh Tavakolipour (Amirkabir University of Technology) [27], we studied with the above tools the asymptotics of eigenvalues and eigenvectors of symmetric positive definite matrices over the field of Puiseux series.

8.3.4 Tropical Systems of Polynomial Equations

Participants: Marianne Akian, Matías Bender, Antoine Bereau, Stéphane Gaubert.

The PhD thesis of Antoine Bereau [71] deals with systems of polynomial equations over tropical semifields. In [39, 46], we established a Nullstellenstatz for sparse tropical polynomial systems. We reduce a polynomial system to a linearized system obtained by an appropriate truncation of the Macaulay matrix. Our approach is inspired by a construction of Canny-Emiris (1993), refined by Sturmfels (1994). It leads to an improved estimate of the truncation degree. We also establish a tropical positivstellensatz, allowing one to decide the containment of tropical basic semialgebraic sets. This method leads to the solution of systems of tropical linear equalities and inequalities, which reduces to mean payoff games [23].

Building on this construction, in collaboration with Yue Ren (University of Durham, UK), we are investigating new efficient methods for computing resultants and solving structured polynomial systems. These systems include cases where the degeneration into tropical forms, such as vertically and horizontally parameterized systems, is well understood [94].

8.3.5 Systems of sparse polynomial equations and convex polytopes

Participants: Matías Bender.

Solving systems of polynomial equations is an intrinsically hard problem. In applications, almost all the systems that we encounter have certain structure, so it is central to take advantage of this to be able to solve bigger inputs. Among the most common structures, we find sparsity: the polynomial are defined by a few set of monomials. If all the polynomial share the same sparsity pattern we say that the system is "unmixed"; otherwise, we call it "mixed". Several strategies had been proposed to deal with sparse polynomial systems, being the more efficients ones the ones to deal with unmixed systems. In our recent paper [18] with Pierre-Jean Spaenlehauer (Inria Nancy), we characterise the conditions under which the specialised strategies developed for unmixed systems can be applied to mixed systems. Our analysis relies on the combinatorics of convex polytopes associated to the input polynomials, i.e., Newton polytopes, and toric geometry. Furthermore, we show that deciding whenever this can be done is a NP-hard problem.

In [31], in collaboration with Laurent Busé (Inria d'Université Côte d'Azur), Carles Checa (University of Copenhagen, Denmark), and Elias Tsigaridas (Inria Paris), we studied mixed sparse systems whose Newton polytopes are Cartesian products of simplices. We showed that the complexity of computing Groebner bases for these systems can be characterized in terms of the multigraded Castelnuovo-Mumford regularity, providing a partial generalization of the milestone work of Bayer and Stillman on homogeneous polynomials [70]. In [32], we use our characterization to introduce a novel method for computing the solutions to projections of these systems onto a subset of variables. This linear algebra-based approach offers new techniques for computing Gröbner bases and obtaining numerical approximations for such projections in single exponential time, but polynomial time relative to the worst-case size of the output.

8.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

8.4.1 Tropicalization of interior point methods and application to complexity

Participants: Xavier Allamigeon, Stéphane Gaubert, Nicolas Vandame.

It is an open question to determine if the theory of self-concordant barriers can provide an interior point method with strongly polynomial complexity in linear programming. In the special case of the logarithmic barrier, it was shown in [60],[7] that the answer is negative.

In a subsequent work [51] with Abdellah Aznag (Columbia University) and Yassine Hamdi (Ecole Polytechnique), we have studied the tropicalization of the central path associated with the entropic barrier studied by Bubeck and Eldan (Proc. Mach. Learn. Research, 2015), i.e., the logarithmic limit of this central path for a parametric family of linear programs defined over the field of Puiseux series. Our main result is that the tropicalization of the entropic central path is a piecewise linear curve which coincides with the tropicalization of the logarithmic central path studied by Allamigeon et al. in [60],[7].

In the work [44], we have now shown that *none* of the self-concordant barrier interior point methods is strongly polynomial. This result is obtained by establishing that, on parametric families of convex optimization problems, the log-limit of the central path degenerates to the same piecewise linear curve, independently of the choice of the barrier function. We also provided an improved counter example, with an explicit linear program that falls in the same class as the Klee–Minty counterexample, *i.e.*, a n-dimensional combinatorial cube, in which the number of iterations is 2^n.

A key tool in this work consists of metric inequalities, controlling the convergence of the log-images of semialgebraic sets to a polyhedral complex (their tropicalization). Explicit convergence bounds have been subsequently established, see §8.4.4 below. These results are presented in the Phd thesis of Nicolas Vandame, defended in December 2024 [128].

In a joint work [43] with Daniel Dadush, Georg Loho, Bento Natura and László Végh, we establish a natural connection between the complexity of interior point methods and that of the simplex method, and deduce combinatorial bounds on the number of iterations. In more details, we introduce a new polynomial-time path-following interior point method where the number of iterations also admits a combinatorial upper bound $O(2^n n^{1.5} \log n)$ for an n-variable linear program in standard form. The number of iterations of our algorithm is at most $O(n^{1.5} \log n)$ times the number of segments of any piecewise linear curve in the wide neighborhood of the central path. In particular, it matches the number of iterations of any path following interior point method up to this polynomial factor. The overall exponential upper bound derives from studying the 'max central path', a piecewise-linear curve with the number of pieces bounded by the total length of $2n$ shadow vertex simplex paths.

8.4.2 Tropical Nash equilibria and complementarity problems

Participants: Xavier Allamigeon, Stéphane Gaubert.

Linear complementarity programming is a generalization of linear programming which encompasses the computation of Nash equilibria for bimatrix games. While the latter problem is PPAD-complete, we show in [53] that the analogue of this problem in tropical algebra can be solved in polynomial time. Moreover, we prove that the Lemke–Howson algorithm carries over the tropical setting and performs a linear number of pivots in the worst case. A consequence of this result is a new class of (classical) bimatrix games for which Nash equilibria computation can be done in polynomial time. This is joint work with Frédéric Meunier (Cermics, ENPC).

8.4.3 Signed Tropicalization of Polars and application to Matrix Cones

Participants: Marianne Akian, Xavier Allamigeon, Stéphane Gaubert.

With Sergey Sergeev (U. Birmingham), we study in [26] the tropical analogue of the notion of polar of a cone over the symmetrized tropical semiring (see for instance [115, 66]). We characterize in particular the tropical polars of sets of nonnegative tropical vectors, and relate them with images by the nonarchimedean valuation of classical polars over real closed nonarchimedean fields. We study in particular cones of matrices, and optimization problems.

8.4.4 Log-limits of semi-algebraic sets

Participants: Xavier Allamigeon, Stéphane Gaubert, Nicolas Vandame.

We study the log-convergence of parametric families of semialgebraic sets to their valuation. We obtain explicit metric estimates under genericity conditions for basic semialgebraic sets. In the general case, we establish a multivariate extension of the Denef–Pas cell decomposition. Using Smale's α-theory, we give a quantitative version of this decomposition that allows us to prove a metric lifting theorem for semialgebraic sets. It yields a constructive bound on the one-sided Hausdorff convergence of the parametric family to its valuation. These results are presented in the PhD thesis of Nicolas Vandame, defended in December 2024 [128].

8.5 A metric-geometry approach to LLM

Participants: Stéphane Gaubert.

In a joint work with Yannis Vlassopoulos (Athena research center), we have been developing a metric geometry approach to large language models. Texts are thought of as elements of a metric space, in which a nonsymmetric distance provides the logarithm of the probability that a text extends another text. We studied several spaces associated to this distance, including alcoved polyhedra (a subclass a tropical polyhedra) and directed tight-spans. This is presented in [34].

8.6 Polynomial optimization and certificates of positivity

Participants: Matías Bender.

The Polynomial Optimization Problem involves optimizing a polynomial objective function over a set defined by polynomial inequalities. A key special case is the Unconstrained one, which focuses on minimizing a multivariate polynomial over the reals. This problem has broad applications across science and engineering and is deeply connected to real algebraic geometry. Certification plays a crucial role in this context, especially through Sum of Squares (SOS) techniques, which provide proofs of feasibility or positivity. However, exact SOS certificates are not always guaranteed to exist, and while some can be constructed under specific conditions—such as for univariate polynomials—the current bounds on their degrees make them largely theoretical.

In recent years, new methods have emerged for solving unconstrained optimization problems and providing exact certificates in special cases, particularly when a polynomial has a finite number of critical points. In [103, 68], different authors explored this scenario, showing that under further assumptions on the structure of critical points, certificates of positivity can be computed with single exponential bounds. In [33], together with Chaoping Zhu and Elias Tsigaridas (Inria Paris), we demonstrate that some of these restrictions can be overcome while maintaining the same complexity. Additionally, we present the first approach that incorporates the sparsity structure of the input polynomial, a common feature in many real-world problems.

8.7 Applications

8.7.1 Performance evaluation of emergency call centers and emergency services

Participants: Xavier Allamigeon, Pascal Capetillo, Stéphane Gaubert, Benjamin Nguyen-Van-Yen, Guillaume Thomas.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization (PFAU, "Plate forme d'appels d'urgence") to handle emergency calls to firemen and policemen in the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [61], [62], [72]. In 2019, with four students of École polytechnique, Céline Moucer, Julia Escribe, Skandère Sahli and Alban Zammit, we performed case studies, showing the improvement brought by the two level filtering procedure.

Moreover, in 2019, this work has been extended to encompass the handling of health emergency calls, with a new collaboration, involving responsibles from the four services of medical emergency aid of Assistance Publique – Hôpitaux de Paris (APHP), i.e., with SAMU75, 92, 93, 94, in the framework of a project coordinated by Dr. Christophe Leroy from APHP. As part of his PhD work, Marin Boyet have developed Petri net models capturing the characteristic of the centers (CRRA) handling emergency calls the SAMU, in order to make dimensioning recommendations. Following this, we have been strongly solicited by APHP during the pandemic of Covid-19 in order to determine crisis dimensioning of SAMU.

In parallel, we have further investigated the theoretical properties of timed Petri nets with preselection and priority routing. We represent the behavior of these systems by piecewise affine dynamical systems. We use tools from the theory of nonexpansive mappings to analyze these systems. We establish an equivalence theorem between priority-free fluid timed Petri nets and semi-Markov decision processes, from which we derive the convergence to a periodic regime and the polynomial-time computability of the throughput. More generally, we develop an approach inspired by tropical geometry, characterizing the congestion phases as the cells of a polyhedral complex. These results are illustrated by the application to the performance evaluation of emergency call centers of SAMU in the Paris area. These results have been published in [37].

In [41], we provided explicit formulæ allowing one to compute the time needed by a call center to return to a stationary state after a bulk of calls. This is based on a turnpike-type theorem for Markov decision processes.

These results are also presented in the Phd thesis [74].

In a current followup work, in the framework of the URGE project (see Section 10.2.3 below) we are extending these models in order to evaluate the dimensioning of emergency departments. This is the object of the PhD work of Pascal Capetillo, which started in Nov. 2023.

8.7.2 Stationary regimes of piecewise linear dynamical systems with priorities

Participants: Xavier Allamigeon, Pascal Capetillo, Stéphane Gaubert.

A key question in the study of Petri nets with priority is the existence of stationary regimes. (See section above for motivations.) The counter equations of these Petri nets, modelling the evolution of events, follow piecewise linear equations that are similar to the dynamic programming equations of semi-Markov decision processes, up to a key new feature: the "probabilities' can take negative values – which entails that the dynamics is no-longer monotone. Although stationary regimes were found (by a case by case analysis) for a number of concrete models, including emergency call centers, it has been an open problem to show that these regimes do exist under general circumstances. Using techniques of topological degree theory, we show in [30] that stationary regimes do exist, independently of the choice of resources, for a broad enough class of systems (including all known examples). This result also extends Kohlberg's theorem (existence of invariant half-lines for nonexpansive piecewise affine maps), relaxing the nonexpansiveness assumption.

8.7.3 Optimal pricing of energy contracts

Participants: Abdellah Bulaich, Stéphane Gaubert.

This work builds on the PhD thesis of Quentin Jacquet [97], which was cosupervised by Stéphane Gaubert, Clémence Alasseur (EDF Labs), and Wim van Ackooij (EDF Labs). It concerns the application of bilevel programming methods to the pricing of electricity contracts. We investigated in [22] a new model of customer's response, based on a quadratic regularization. We showed that this model has qualitative properties and a realism similar to the classical models based on the logit-response, while being amenable to mathematical programming and polyhedral techniques, and so to exact solutions, via a reduction to quadratic complementary problems. An application to a set of instances representative of French electricity contracts was also developed in [22].

In [21], we developed a model representing the response of a population of customers to dynamic price offers. This lead to a mean-field Markov decision model, with ergodic cost. We showed in particular the optimality of pricing policies characterized by periodic discounts.

The preprint [50] develops a mean field game model to model an incentive pricing scheme, in which agents compete to get the best reward.

The conference article [45] developed a method to compute an optimal menu of contracts of a prescribed cardinality, optimizing the income of a provider, taking into account the agent's characteristics.

This work was subsequently pursued within the "Defi EDF Inria", with a collaboration between Wim van Ackooij, Luce Brotcorne (Inria Lille) and S. Gaubert, leading to the hiring in internship of Abdellah

Bulaich-Mehamdi at the spring 2024, pursuing in CIFRE PhD in 2025. We further investigated the problem of design of a menu of contracts of a prescribed cardinality, in particular, we related its formulation in terms of the best approximation of a convex function by a polyhedral function with a prescribed number of facets to the problem of optimal quantization of probability measures. A first account of these results was presented at the conference “PGMO Days 2024” at EDF Labs, Saclay.

8.7.4 Aggregated models of flexibilities in long term investment models

Participants: Stéphane Gaubert.

Within the framework of the “Defi EDF Inria”, we started a joint project with the Polaris team of Inria Rhône-Alpes (Bruno Gaujal and Nicolas Gast) and with Olivier Beaude and Juliette Lesturgie, from EDF Labs, on the development of aggregated models of flexibilities of customers in the electricity field. Examples of flexibilities are given by parks of electric vehicles, or hydroelectric resources. These aggregated models are used to optimize long term investments. This is the object of the starting PhD thesis of Hélène Arvis, jointly supervised with Polaris (main supervisor: Nicolas Gast). An internship preparatory to the PhD took place during the last six months of 2024, Hélène Arvis is now starting her CIFRE PhD.

8.7.5 Simulation and optimization of robust strategies for spectrum auctions

Participants: Marianne Akian, Stéphane Gaubert, Jad Zeroual.

In order to deploy new 5G mobile networks, telecommunication operators must acquire new frequency bands (called spectrum). The acquisition of these frequencies is done, in each country, through participation in auctions whose rules are specific and dictated by the licences' owners (usually states). The aim of the PhD thesis of Jad Zeroual, supervised by Marianne Akian, Stéphane Gaubert, Aurélien Bechler (Orange Labs) and Mathieu Chardy (Orange Labs), is to develop mathematical models allowing one to facilitate the understanding and participation in licensed spectrum auctions, and in particular to compute optimal acquisition strategies of the desired amount of spectrum.

In a first work [25], we studied a model of auction representative of the 5G auction in France. We determine the optimal strategy of a bidder, assuming that the valuations of competitors are unknown to this bidder and that competitors adopt the straightforward bidding strategy. The model is based on a Partially Observable Markov Decision Process (POMDP). We show, under some assumptions on the probability distributions of the valuations of the competitors, that this POMDP admits a concise statistics, avoiding the solution of a dynamic programming equation in the space of beliefs. The results are also illustrated by numerical experiments, comparing the value of the bidder with the value of a perfectly informed one. The paper [25] obtained the best paper award at the conference NETGCOOP'2024.

9 Bilateral contracts and grants with industry

9.1 Bilateral contracts with industry

Participants: Marianne Akian, Stéphane Gaubert, Abdellah Bulaich Mehamdi, Jad Zeroual.

- *Simulation and optimization of robust strategies for spectrum auctions.* Collaboration of Marianne Akian and Stéphane Gaubert with Aurélien Bechler and Mathieu Chardy from Orange Labs, for the supervision of the Phd work of Jad Zeroual (CIFRE PhD from January 9, 2024, directed by Marianne Akian).

- *Optimal pricing of energy contracts.* Collaboration of Stéphane Gaubert with Luce Brotcorne (Inria Lille, INOCS team), Wim Van Ackooij and Quentin Jacquet, from EDF Labs. This is part of the "DEFI EDF-inria". This supports the CIFRE Phd Thesis of Adbellah Bulaich-Mehamdi.
- *Aggregated models of flexibilities in long term investment models.* Collaboration of Stéphane Gaubert with Bruno Gaujal and Nicolas Gast (Inria Rhône-Alpes, Polaris team) and Olivier Beaude and Juliette Lesturgie, from EDF Labs. This is part of the "DEFI EDF-inria". This supports the CIFRE Phd Thesis of Hélène Arvis.

10 Partnerships and cooperations

10.1 International research visitors

10.1.1 Visits of international scientists

Other international visits to the team

Carles Checa

Status Post-Doc

Institution of origin: University of Copenhagen

Country: Denmark

Dates: 25-29 November.

Context of the visit: Collaboration with Matías Bender and Elias Tsigaridas (Inria Paris) which lead to the paper [32].

Mobility program/type of mobility: research stay

Philipp di Dio

Status Professor

Institution of origin: Universität Konstanz

Country: Germany

Dates: 2-6 December.

Context of the visit: Collaboration with Matías Bender and Elias Tsigaridas (Inria Paris) on polynomial optimization.

Mobility program/type of mobility: research stay

Michael Joswig

Status Professor

Institution of origin: TU Berlin

Country: Germany

Dates: 4-15 Nov.

Context of the visit: Joint invitation of CMAP, CMLS and Inria.

Mobility program/type of mobility: research stay

Yue Ren

Status Professor

Institution of origin: Durham University

Country: United Kingdom

Dates: 21-25 October.

Context of the visit: Talk at the Tropical day and collaboration with Marianne Akian, Matías Bender, and Stéphane Gaubert on tropical methods to solve polynomial systems.

Mobility program/type of mobility: research stay

Hanieh Tavakolipour

Status Post-Doc

Institution of origin: Amirkabir University

Country: Iran

Dates: 1-5 July.

Context of the visit: Collaboration with Marianne Akian and Stéphane Gaubert which started in 2020, when Hanieh Tavakolipour was postdoc in the team.

Mobility program/type of mobility: research stay

10.2 National initiatives

10.2.1 ANR

Participants: Marianne Akian, Xavier Allamigeon, Cormac Walsh, Stéphane Gaubert.

- Project ANR HilbertXField ("Géométries de Hilbert sur tout corps valué"). ANR leader: Antonin Guilloux. Partners: IMJ-PRG/OURAGAN (Sorbonne Université, pole leader Antonin Guilloux), CMAP/TROPICAL (Inria, pole leader: Cormac Walsh), Institut Fourier (Grenoble, pole leader Anne Parreau).

10.2.2 Programme Gaspard Monge pour l'optimisation, la recherche opérationnelle et leurs interactions avec les sciences des données

Participants: Matías Bender.

- SOAP - Sparsity in Optimization via Algebra and Polynomials, with Elias Tsigaridas (IMJ), project funded by FMJH within the PGMO programme.

10.2.3 Joint INRIA & AP-HP Bernoulli lab project: "URGE"

- The project URGE (Analyse des parcours patients aux URgences et optimisation des prises en charGE), started at the fall 2022, in the framework of the joint INRIA & AP-HP Bernoulli lab. The goal of the project is to develop modelling, simulation, performance analysis, and visualization tools, in order to help physicians to optimize the staffing of emergency services. This collaborative project, of four years, involves the Tropical and Aviz teams from INRIA Saclay, the Dyogene team from INRIA Paris, and the Fédération Hospitalo-Universitaire (FHU) / IMPEC Improving Emergency Care, AP-HP / Sorbonne Université / INSERM. The project is led by X. Allamigeon (Tropical) and Y. Yordanov (AP-HP, Saint-Antoine) and involves S. Gaubert, B. Nguyen (Tropical), Ch. Fricker (Dyogene), J.D. Fekete (Aviz).

10.3 Public policy support

10.3.1 Performance evaluation of emergency call centers

Participants: Xavier Allamigeon, Stéphane Gaubert, Benjamin Nguyen-Van-Yen, Guillaume Thomas
.

Collaboration with Préfecture de Police on the performance evaluation of the call center "PFAU" ("Plate forme d'appels d'urgence"), to handle emergency calls to firemen and policemen in the Paris area. See Section 8.7.1 for more information.

11 Dissemination

11.1 Promoting scientific activities

11.1.1 Scientific events: organisation

General chair, scientific chair

- Marianne Akian was one of the 3 directors of the Conference VARANA 2024 (Variational analysis and applications), which was the 77th Course of the International School of Mathematics «Guido Stampacchia», 1-7 Sep 2024 Erice (Italy), see the web site.
- Matías Bender organised Tropical day, a one-day international workshop on tropical methods and polynomial systems at CMAP, École Polytechnique. October 2024.
- Stéphane Gaubert is the adjunct coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorhip program, operated by Fondation Mathématique Jacques Hadamard, supported by EDF, see Pgmo site.

Member of the organizing committees

- Stéphane Gaubert co-organizes the "Séminaire Parisien d'Optimisation" at Institut Henri Poincaré.

11.1.2 Scientific events: selection

Chair of conference program committees

- Stéphane Gaubert, co-chair of the PGMO Days 2024, EDF Labs, Nov.

Member of the conference program committees

- Marianne Akian, member of the Short Communications Committee of ISSAC'2024.

11.1.3 Journal

Member of the editorial boards

- Stéphane Gaubert: member of the editorial board of Journal of Dynamics and Games, Linear and Multilinear Algebra, RAIRO, Springer-SMAI book series.

11.1.4 Invited talks

- Marianne Akian:
  - Talk in the minisymposium "Tropical Methods for Numerical Linear Algebra" at SIAM Conference on Applied Linear Algebra (LA24), May 13–17 , 2024, Paris, on "Spectral elements of positive definite matrices over symmetrized tropical algebra" (work with Stéphane Gaubert, Dariush Kiani, and Hanieh Tavakolipour).
  - Talk at the Dagstuhl seminar on Stochastic Games, on "Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration" (Jun 02 – Jun 07, 2024).
  - Talk in the special session "Positive operators and their dynamics" at the 35th International Workshop on Operator Theory and its Applications (IWOTA 2024), University of Kent, Canterbury, UK, on "Escape rate games and competitive spectral radi" (work with Stéphane Gaubert and Loïc Marchesini.
  - Talk in the minisymposium "Solution of Hamilton-Jacobi Equations" at the 26th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2024) 19-23 August 2024, Cambridge, UK, on "Escape Rate Games" (work with Stéphane Gaubert and Loïc Marchesini.
- Matías Bender:
  - Talk at "Back to the roots" seminar of the Department of Electrical Engineering, KU Leuven, Belgium. March 2024.
  - Invitation to the BIRS program Representation Theory and Topological Data Analysis at Banff, Canada. April 7 - 12, 2024.
  - Invitation to the BIRS program Positive Solutions of Polynomial Systems Arising from Real-life Applications at Granada, Spain. May 19 - 24, 2024.
  - Talk at minisymposium "Numerical Linear Algebra Algorithms to Solve (Multivariate) Polynomial Systems" at the 2024 SIAM Conference on Applied Linear Algebra. Paris. May 2024.
  - Talk at the "International Workshop on Recent Advances in MAthematics (RAMA 2024)", Hangzhou, China. November 2024.
- Antoine Béreau:
  - Talk in the special session "Polyhedral geometry and combinatorics" at International Congress on Mathematical Software 2024 (ICMS), Durham U., UK, 22-25 July, 2024, on "Eigenvalue Methods for Sparse Tropical Polynomial Systems" (work with Marianne Akian and Stéphane Gaubert).
- Stéphane Gaubert:
  - Plenary talk at the 35th International Workshop on Operator Theory and its Applications (IWOTA 2024), University of Kent, Canterbury, UK, on the topic of "Tropical geometry, operators and games". The video is available here.
  - Plenary talk at the "CaLISTA Workshop Geometry-Informed Machine Learning", on "Tropical convexity and games: regression, separation, and beyond", École des Mines, Paris, Sep 2-5, 2024.

- Talk at the Dagstuhl seminar on Stochastic Games, on “Tropical geometry, linear programming and games” (Jun 02 – Jun 07, 2024).
- Talk at the Dagstuhl seminar on Network calculus, on “Tropical geometry and piecewise-linear dynamical systems applied to the dimensioning of emergency call centers” (April 02 – 04, 2024).
- Talk in the minisymposium “Recent Trends in Tensors and Quantum Information” at SIAM Conference on Applied Linear Algebra (LA24), May 13–17 , 2024, Paris, on “Polynomial Computability of Spectral Radius of Nonnegative Weakly Irreducible Tensors”.

11.1.5 Research administration

Inria research administration

- Marianne Akian: Elected member of Inria’s Scientific Board.
- Matìas Bender: Member of the scientific committee of INRIA Saclay.

Other research administration

- Marianne Akian: “Secrétaire Générale” of SMAI since July 2023.

11.2 Teaching - Supervision - Juries

11.2.1 Teaching

- Marianne Akian
  - Course “Markov decision processes: dynamic programming and applications” joint between (3rd year of) ENSTA and M2 “Mathématiques et Applications”, U. Paris Saclay, “Optimization”, 30 hours.
- Xavier Allamigeon
  - Petites classes et encadrement d’enseignements d’approfondissement de Recherche Opérationnelle en troisième année à l’École Polytechnique (programme d’approfondissement de Mathématiques Appliquées) (niveau M1).
  - Cours “Theoretical Aspects of Linear Programming” du M2 “Optimisation” de l’Université Paris Saclay.
- Matías Bender
  - Exercises classes for the first and second years of Bachelor program of École Polytechnique in the framework of a “Chargé d’ensegnement”.
- Antoine Béreau
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Amanda Bigel
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Pascal Capetillo
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- Yiyuan Chen

  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.

- Stéphane Gaubert
  - Co-head of the Master “Optimization” of University Paris-Saclay and IPP.
  - Head for IP Paris of the “Mention” (Master Programme) “Mathématiques et Applications” jointly operated by University Paris Saclay and IP Paris (since Sep. 2024).
  - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
  - Course “Algèbre tropicale pour le contrôle optimal et les jeux” of “Contrôle, Optimisation et Calcul des Variations” (COCV) of M2 “Mathématiques et Applications” of Sorbonne University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [73].

- Jonathan Hornewall
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.

11.2.2 Supervision

- PhD in progress: Quentin Canu, registered at Univ. Paris Saclay since October 2020, thesis supervisor: Georges Gonthier (INRIA), cosupervision: Xavier Allamigeon and Pierre-Yves Strub (LIX).
- PhD of Antoine Bereau, registered at IPP (EDMH), started in September 2021, defended in November 2024. Thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD of Nicolas Vandame, registered at IPP (EDMH), started in September 2021, defended in December 2024, supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Amanda Bigel, registered at IPP (EDMH) since September 2022, main thesis supervisors: Cormac Walsh et Constantin Vernicos, thesis supervisor: Stéphane Gaubert.
- PhD in progress: Loïc Marchesini, registered at IPP (EDMH), since September 2023. Thesis supervisor Marianne Akian, co-supervised by Stéphane Gaubert.
- PhD in progress: Pascal Capetillo, registered at IPP (EDMH), since November 2023. Thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Jonathan Hornewall, registered at Marne-la-vallée, ENPC (maths and STIC), since November 2023. Thesis supervisor: Vincent Leclere, cosupervision: Stéphane Gaubert.
- PhD in progress: Jad Zeroual, registered at IPP (EDMH), since January 2024. Thesis supervisor Marianne Akian, co-supervised by Stéphane Gaubert, Aurélien Bechler (Orange Labs) and Mathieu Chardy (Orange Labs).
- PhD in progress: Yiyuan Chen, registered at IPP (EDMH), since September 2024. Thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.

11.2.3 Juries

- Marianne Akian
  - Chair of the Inria junior researchers hiring committee at Inria Bordeaux.
  - Jury of the PhD thesis of Christelle Kozaily, Université de Rennes, Oct. 29, 2024.

- Stéphane Gaubert

- Jury of the PhD thesis of Antoine Béreau, Ecole polytechnique, November 2024.
- Jury of the PhD thesis of Nicolas Vandame, Ecole polytechnique, December 2024.
- Jury (president) of the PhD thesis of Lucas Gierczak, Ecole polytechnique, December 2024.
- Jury of the PhD thesis of Maël Bompais, Université Paris-Saclay, December 2024.
- Jury (reviewer) of the PhD thesis of Germano Schafaschek, Fakultät IV - Electrotechnik und Informatik, TU-Berlin, December 2024.

11.3 Conferences, Seminars

- Marianne Akian
  - Talk in the Workshop "Dynamic Games", Fréjus, May 25-28, 2024, on "Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration".
- Antoine Béreau
  - Talk at A Tropical Day (Oct. 21) on "Eigenvalue methods for sparse tropical polynomial systems".
- Abdellah Bulaich Mehamdi
  - Talk at the PGMODAYS 2024 (Nov. 19-20) on "Rochet-Choné Model, Bi-Level Optimization, and Quantization for Electricity Pricing".
- Pascal Capetillo
  - Talk at the PGMODAYS 2024 (Nov. 19-20) on "Stationary Regimes of Piecewise Linear Dynamical Systems with Priorities".
- Stéphane Gaubert
  - Talk in the Workshop "Dynamic Games", Fréjus, May 25-28, 2024, on "Game of escape rate".
- Jonathan Hornewall
  - Talk at the PGMODAYS 2024 (Nov. 19-20) on "The Stochastic Central Path : Sampling the Optimal Solution of a Convex Program with Random Parameters".
- Loic Marchesini
  - Poster at Journées SMAI MODE 2024 (March 27-29) on "Jeux de taux de fuite et jeux de multiplication matricielle".
  - Talk at CJC 2024 (Lyon, Oct. 28-30).
  - Talk at the "Junior Game Theory Seminar" (IHP, Nov. 18).
  - Talk at the PGMODAYS 2024 (Nov. 19-20) on "Escape Rate Games and Competitive Spectral Radii".
- Jad Zeroual
  - Poster at the "50 years of CMAP" (Sep. 11-13).
  - Talk at the conference NETGCOOP 2024 (Lille, Oct. 9-11) on "Optimal Strategy against Straightforward Bidding in Clock Auctions".
  - Talk at the PGMODAYS 2024 (Nov. 19-20) on "Optimal Strategy against Straightforward Bidding in Clock Auctions".

12 Scientific production

12.1 Major publications

[1] M. Akian, S. Gaubert and A. Guterman. 'Tropical polyhedra are equivalent to mean payoff games'. In: *Internat. J. Algebra Comput.* 22.1 (2012), pp. 1250001, 43. DOI: 10.1142/S0218196711006674. eprint: 0912.2462. URL: http://dx.doi.org/10.1142/S0218196711006674 (cit. on pp. 3, 5).

[2] M. Akian, S. Gaubert and R. Nussbaum. 'Uniqueness of the fixed point of nonexpansive semidifferentiable maps'. In: *Transactions of the American Mathematical Society* 368.2 (Feb. 2016). Also arXiv:1201.1536. DOI: 10.1090/S0002-9947-2015-06413-7. URL: https://hal.inria.fr/hal-00783682 (cit. on pp. 3–5).

[3] M. Akian, S. Gaubert and R. Bapat. 'Non-archimedean valuations of eigenvalues of matrix polynomials'. In: *Linear Algebra and its Applications* 498 (June 2016). Also arXiv:1601.00438, pp. 592–627. DOI: 10.1016/j.laa.2016.02.036. URL: https://hal.inria.fr/hal-01251803.

[4] M. Akian, S. Gaubert and C. Walsh. 'The max-plus Martin boundary'. In: *Doc. Math.* 14 (2009), pp. 195–240 (cit. on p. 4).

[5] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'Combinatorial simplex algorithms can solve mean payoff games'. In: *SIAM J. Opt.* 24.4 (2015), pp. 2096–2117. eprint: 1309.5925.

[6] X. Allamigeon, S. Gaubert, E. Goubault, S. Putot and N. Stott. 'A scalable algebraic method to infer quadratic invariants of switched systems'. In: *Proceedings of the International Conference on Embedded Software (EMSOFT)*. Best paper award. The extended version of this conference article appeared in *ACM Trans. Embed. Comput. Syst., 15(4):69:1–69:20, September 2016*. 2015 (cit. on pp. 3, 6).

[7] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'What Tropical Geometry Tells Us about the Complexity of Linear Programming'. In: *SIAM Review* 63.1 (4th Feb. 2021), pp. 123–164. DOI: 10.1137/20M1380211. URL: https://hal.inria.fr/hal-03505719 (cit. on pp. 5, 12).

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[10] S. Gaubert and T. Lepoutre. 'Discrete limit and monotonicity properties of the Floquet eigenvalue in an age structured cell division cycle model'. In: *J. Math. Biol.* (2015). DOI: 10.1007/s00285-015-0874-3. eprint: 1301.2151. URL: http://dx.doi.org/10.1007/s00285-015-0874-3 (cit. on pp. 3, 4).

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12.2 Publications of the year

International journals

[13] M. Akian, J.-P. Chancelier and B. Tran. 'A stochastic algorithm for deterministic multistage optimization problems'. In: *Annals of Operations Research* 345 (13th Jan. 2025), pp. 1–38. DOI: 10.1007/s10479-024-06153-8. URL: https://inria.hal.science/hal-01964189 (cit. on p. 8).

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[15] M. Akian, S. Gaubert and S. Liu. 'A Multi-Level Fast-Marching Method For The Minimum Time Problem'. In: *SIAM Journal on Control and Optimization* 62.6 (2024), pp. 2963–2991. DOI: 10.1137/23M1563657. URL: https://inria.hal.science/hal-04365665 (cit. on p. 8).

[16] X. Allamigeon, S. Gaubert, R. Katz and M. Skomra. 'Universal complexity bounds based on value iteration for stochastic mean payoff games and entropy games'. In: *Information and Computation* 302 (Jan. 2025), p. 105236. DOI: 10.1016/j.ic.2024.105236. URL: https://inria.hal.science/hal-04859637.

[17] P.-C. Aubin-Frankowski and S. Gaubert. 'Tropical reproducing kernels and optimization'. In: *Integral Equations and Operator Theory* 96.19 (2024). URL: https://hal.science/hal-03588622.

[18] M. R. Bender and P.-J. Spaenlehauer. 'Dimension results for extremal-generic polynomial systems over complete toric varieties'. In: *Journal of Algebra* 646 (2024), pp. 156–182. DOI: 10.1016/j.jalgebra.2024.01.029. URL: https://inria.hal.science/hal-04102564 (cit. on p. 12).

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[20] M. Forcier, S. Gaubert and V. Leclère. 'Exact quantization of multistage stochastic linear problems'. In: *SIAM Journal on Optimization* 34.1 (5th Feb. 2024), pp. 533–562. DOI: 10.1137/22M1508005. URL: https://inria.hal.science/hal-03504876.

[21] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Ergodic control of a heterogeneous population and application to electricity pricing'. In: *IEEE Transactions on Automatic Control* (2025). URL: https://hal.science/hal-03629189. In press (cit. on p. 15).

[22] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Quadratic regularization of bilevel pricing problems and application to electricity retail markets'. In: *European Journal of Operational Research* 313.3 (Mar. 2024), pp. 841–857. DOI: 10.1016/j.ejor.2023.05.006. URL: https://inria.hal.science/hal-03504874 (cit. on p. 15).

International peer-reviewed conferences

[23] M. Akian, A. Béreau and S. Gaubert. 'Eigenvalue Methods for Sparse Tropical Polynomial Systems'. In: *Lecture Notes in Computer Science*. ICMS 2024 - International Congress on Mathematical Software. Durham, United Kingdom: Springer, 2024. URL: https://inria.hal.science/hal-04575772 (cit. on p. 11).

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[25] J. Zeroual, M. Akian, A. Bechler, M. Chardy and S. Gaubert. 'Optimal Strategy Against Straightforward Bidding in Clock Auctions'. In: NETGCOOP 2024 - Network Games, Artificial Intelligence, Control and Optimization. Vol. 15185. Lecture Notes in Computer Science. Lille, France: Springer Nature Switzerland, 21st Jan. 2025, pp. 83–93. DOI: 10.1007/978-3-031-78600-6_8. URL: https://inria.hal.science/hal-04907159 (cit. on p. 16).

Reports & preprints

[26] M. Akian, X. Allamigeon, S. Gaubert and S. Sergeev. *Signed tropicalization of polar cones*. Feb. 2024. URL: https://inria.hal.science/hal-04365674 (cit. on p. 13).

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[34] S. Gaubert and Y. Vlassopoulos. *Directed Metric Structures arising in Large Language Models*. 20th May 2024. URL: https://inria.hal.science/hal-04582412 (cit. on p. 14).

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[36] B. Lemmens and C. Walsh. *Distance-preserving maps between bounded symmetric domains*. 2024. URL: https://hal.science/hal-04911597 (cit. on p. 9).

12.3 Cited publications

[37] X. Allamigeon, M. Boyet and S. Gaubert. 'Piecewise Affine Dynamical Models of Timed Petri Nets – Application to Emergency Call Centers'. In: *Fundamenta Informaticae* 183.3-4 (2021), pp. 169–201. URL: https://hal.archives-ouvertes.fr/hal-02550006 (cit. on p. 15).

[38] X. Allamigeon, S. Gaubert and M. Skomra. 'Tropical spectrahedra'. In: *Discrete and Computational Geometry* 63 (Feb. 2020), pp. 507–548. DOI: 10.1007/s00454-020-00176-1. URL: https://hal.inria.fr/hal-01422639 (cit. on pp. 5, 11).

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[40] M. Akian, J.-P. Chancelier, L. Pascal and B. Tran. 'Tropical numerical methods for solving stochastic control problems'. In: MTNS 2022 - 25th International Symposium on Mathematical Theory of Networks and Systems. Bayreuth (DE), Germany, 12th Sept. 2022. URL: https://hal.inria.fr/hal-03944216 (cit. on p. 8).

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