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2025 *Activity Report*

RESEARCH CENTRE: **Inria Centre at Université Côte d'Azur**

Project-Team

FACTAS

Functional Analysis for ConcepTion and Assessment of Systems

Project-Team FACTAS

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Each year, Inria research teams publish an Activity Report presenting their work and results over the reporting period. These reports follow a common structure, with some optional sections depending on the specific team. They typically begin by outlining the overall objectives and research programme, including the main research themes, goals, and methodological approaches. They also describe the application domains targeted by the team, highlighting the scientific or societal contexts in which their work is situated. The reports then present the highlights of the year, covering major scientific achievements, software developments, or teaching contributions. When relevant, they include sections on software, platforms, and open data, detailing the tools developed and how they are shared. A substantial part is dedicated to new results, where scientific contributions are described in detail, often with subsections specifying participants and associated keywords. Finally, the Activity Report addresses funding, contracts, partnerships, and collaborations at various levels, from industrial agreements to international cooperations. It also covers dissemination and teaching activities, such as participation in scientific events, outreach, and supervision. The document concludes with a presentation of scientific production, including major publications and those produced during the year.

Keywords

Computer sciences and digital sciences

A6.1.1. – Continuous Modeling (PDE, ODE)

A6.2.1. – Numerical analysis of PDE and ODE

A6.2.5. – Numerical Linear Algebra

A6.2.6. – Optimization

A6.3.1. – Inverse problems

A6.3.3. – Data processing

A6.3.4. – Model reduction

A6.3.5. – Uncertainty Quantification

A6.4.4. – Stability and Stabilization

A6.5.4. – Waves

A8.2. – Optimization

A8.3. – Geometry, Topology

A8.4. – Computer Algebra

A8.10. – Computer arithmetic

Other research topics and application domains

B2.6.1. – Brain imaging

B2.8. – Sports, performance, motor skills

B3.1. – Sustainable development

B3.3. – Geosciences

B5.4. – Microelectronics

B8.4. – Security and personal assistance

B9.1. – Education

B9.5.5. – Mechanics

Contents

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2 Overall objectives

The team develops constructive function-theoretic approaches to inverse problems arising in modeling and design, in particular for electro-magnetic systems as well as in the analysis of certain classes of signals.

Data typically consist of measurements or desired behaviors. The general thread is to approximate them by families of solutions to the equations governing the underlying system. This leads us to consider various interpolation, extrapolation, and approximation problems in classes of rational and meromorphic functions, harmonic gradients, or solutions to more general elliptic partial differential equations (PDE), in connection with inverse potential problems. A recurring difficulty is to control the singularities of the approximants.

The mathematical tools pertain to complex, functional analysis, harmonic analysis, approximation theory, operator theory, potential theory, system theory, differential topology, optimization and computer algebra.

Targeted applications mostly concern non-destructive control from measurements of the potential or the field in medical engineering (source recovery in magneto/electro-encephalography), paleo-magnetism (determining the magnetization of rock samples), and more recently obstacle identification (finding electrical characteristics of an object) as well as inverse problems in orthopedic surgery. For all of these, an endeavor of the team is to develop algorithms resulting in dedicated software.

3 Research program

Within the extensive field of inverse problems, much of the research by Factas deals with reconstructing solutions to classical PDE in dimension 2 or 3 along with their singularities, granted some knowledge of their behavior on part of the domain or of its boundary.

Such problems are severely ill-posed (in the sense of Hadamard): they may have no solution (whenever data are corrupted, since the underlying forward operator may only have dense range), several solutions (non-uniqueness, as the forward operator could be non-injective), and even in situations where there exists a unique solution (whence the forward operator is invertible), they suffer from instability (lack of continuity of the inverse operator). Their resolution thus requires regularizing assumptions or regularization processes, in order to set up well-posed problems and to derive efficient algorithms that furnish suitable approximated solutions.

The considered linear elliptic PDE are related to the Maxwell and wave equations, particularly in the quasi-static or time harmonic regime. This involves in particular Laplace, Poisson and conductivity equations, in which the source term often appears in divergence form. However, the Helmholtz equation also comes up as a formulation of the wave equation in the monochromatic regime.

The gist of our approach is to approximate the data by actual solutions of these PDE, assumed to lie in appropriate function spaces. This differs from standard approaches to inverse problems, where descent algorithms are applied to integration schemes of the direct problem; in such methods, it is the equation which gets approximated (in fact: discretized). This also naturally leads us to study convergent algorithms to approximate solutions of such infinite-dimensional optimization problems by solutions to finite-dimensional ones.

3.1 Elliptic PDEs and operators

Inverse problems studied by Factas involve systems governed by an equation of the form $\mathcal{L}\phi = \Psi(m)$, where $\mathcal{L}$ is an elliptic partial differential operator and $\Psi(m)$ a source term depending on some unknown quantity m. The data consist of incomplete measurements of the potential ϕ or its gradient (the field) in a portion of space, away from the support of the source.

3.1.1 Inverse problems of Cauchy type

Laplace equation in dimension 2. Here, as in the next section, we are concerned with the simplest case where $\mathcal{L}\phi = \Delta\phi = 0$ (without any source term, actually with $m = 0$) in some planar domain $\Omega \subset \mathbb{R}^2$, with Δ to indicate the Euclidean Laplacian. The given data consist in measurements of ϕ and its normal derivative on a subset $E \subset \partial\Omega$ of the domain's boundary, assuming they are somewhat regular; say, they should at least belong to $L^p(E)$ for some $p \geq 1$. The aim is to recover the harmonic function ϕ from partial knowledge of the Dirichlet-Neumann data, which is a classical boundary value problem. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, conjugate-gradients of harmonic functions become holomorphic functions. More precisely, whenever ϕ is harmonic in a domain Ω, it admits a conjugate harmonic function $\tilde{\phi}$ such that, thanks to the Cauchy-Riemann equations, the function $f = \phi + i\,\tilde{\phi}$ is holomorphic in Ω; that is: $\bar{\partial} f = 0$. This framework was first advocated in [57] and subsequently received considerable attention. Therefore, reconstructing a function harmonic in a plane domain Ω when Dirichlet-Neumann boundary conditions are already known on a subset $E \subset \partial\Omega$ is equivalent to recover a holomorphic function in Ω from its boundary values on E. It makes good sense in holomorphic Hardy spaces where functions are entirely determined by their values on boundary subsets of positive linear measure, which is the framework for Problem (P) below, for simply-connected smooth enough domains Ω conformally equivalent to the unit disk $\mathbb{D}$.

Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. We denote by H^p the Hardy space of $\mathbb{D}$ with exponent p, which is the closure of polynomials in $L^p(\mathbb{T})$-norm if $1 \leq p < \infty$ and the space of bounded holomorphic functions in $\mathbb{D}$ if $p = \infty$. Functions in H^p have well-defined boundary values in $L^p(\mathbb{T})$, which makes it possible to speak of (traces of) analytic functions on the boundary. To find an analytic function g in $\mathbb{D}$ matching some measured values f approximately on a subset K of $\mathbb{T}$, with K and $\mathbb{T} \setminus K$ of positive Lebesgue measure, we formulate the best constrained approximation problem (or bounded extremal problem "BEP").

(P) Let $1 \leq p \leq \infty$, $f \in L^p(K)$, $w \in L^p(\mathbb{T} \setminus K)$ and $M > 0$; find a function $g \in H^p$ such that $\|g - w\|_{L^p(\mathbb{T}\setminus K)} \leq M$ and $g - f$ is of minimal norm in $L^p(K)$ under this constraint.

There, w is a reference behavior capturing *a priori* assumptions on the solution off K (if known; otherwise, w can be set to 0), while M is some admissible deviation thereof. The value of p reflects the assumptions made on the given data. As shown in [37, 40, 44], the solution to this well-posed convex infinite-dimensional optimization problem can be obtained when $p \neq 1$ upon iterating with respect to a Lagrange parameter the solution to spectral equations for appropriate Hankel and Toeplitz operators[1]. These spectral equations involve the solution to the special case $K = \mathbb{T}$ of (P), which is a standard extremal problem [59].

In the Hilbertian framework $p = 2$, whenever f does not belong to the approximant set, problem (P) rephrases as the following Tikhonov regularized problem.

> Let $f \in L^2(K)$, $w \in L^2(\mathbb{T} \setminus K)$ and $\lambda > 0$.
> Find a function $g \in H^2$ that minimizes $\|g - f\|^2_{L^2(K)} + \lambda \|g - w\|^2_{L^2(\mathbb{T}\setminus K)}$.

Note that the Lagrange parameter λ is uniquely determined by the constraint $\|g - w\|_{L^p(\mathbb{T}\setminus K)} = M$. The numerical resolution of (P) is performed by expanding the functions under study in the Fourier basis.

Problem (P) also allows one to formulate the recovery of the so-called Robin coefficient on part of the boundary, see [28, 49]. Various modifications of (P) can be tailored to meet specific needs. For instance, one can also impose bounds on the real or imaginary part of $g - w$ on $\mathbb{T} \setminus K$, together with prescribed pointwise values in $\mathbb{D}$ [66], which is useful when considering Dirichlet-Neumann problems. The analog of Problem (P) on an annulus, K being now a subset of the outer boundary, can be seen as a means to recover a harmonic function on the inner boundary from Dirichlet-Neumann data on K [65].

These considerations make it clear how to state similar problems in higher dimensions and for more general operators than the Laplacian, provided solutions are essentially determined by the trace of their gradient on part of the boundary which is the case for sufficiently smooth elliptic equations provided that K has some interior in $\partial\Omega$[2] [28, 85].

Laplace equation in dimension 3. Though originally considered in dimension 2, Problem (P) carries over naturally to higher dimensions where analytic functions get replaced by gradients of harmonic functions. Namely, for $n > 2$, given some open set $\Omega \subset \mathbb{R}^n$ and some $\mathbb{R}^n$-valued vector field F on an open subset O of the boundary of Ω, we seek a harmonic function in Ω (where $\Delta\phi = 0$) whose gradient is close to F on O. This is another subject investigated by Factas, for the recovery of a harmonic function (up to an additive constant) in a ball or a half-space from partial knowledge of its gradient in Ω, with $\nabla\phi = F$ known on O. The question is significantly more difficult than its 2-D counterpart considered in the above paragraph, due mainly to the lack of multiplicative structure for harmonic gradients. Still, substantial progress has been made over the last years using methods of harmonic analysis and operator theory.

When Ω is a ball or a half-space, a substitute for holomorphic Hardy spaces is provided by the Stein-Weiss Hardy spaces[3] of harmonic gradients [82].

On the unit ball $\mathbb{B} \subset \mathbb{R}^n$, the analog of Problem (P) is Problem (P_n):

> (P_n) Let $1 \leq p \leq \infty$. Fix Γ an open subset of the unit sphere $\mathbb{S} \subset \mathbb{R}^n$.
> Let further $F \in L^p(\Gamma)$ and $W \in L^p(\mathbb{S} \setminus \Gamma)$ be $\mathbb{R}^n$-valued vector fields.
> Given $M > 0$, find a harmonic gradient $G \in H^p(\mathbb{B})$ such that $\|G - W\|_{L^p(\mathbb{S}\setminus\Gamma)} \leq M$ and $G - F$ is of minimal norm in $L^p(\Gamma)$ under this constraint.

When $p = 2$, the BEP (P_n) was solved in [26] as well as its analog on a shell, when the tangent component of F is a gradient (when Γ is Lipschitz-smooth, the general case follows easily from this). The solution extends the work in [37] to the 3-D case, using a generalization of Toeplitz operators and expansions on the spherical harmonic basis. An important ingredient is a refinement of the Hodge decomposition, that we call the Hardy-Hodge decomposition, see Section 3.2.1.

[1] Under the assumption $f \in H^\infty + C(\mathbb{T}) \subset L^\infty(\mathbb{T})$ for $p = \infty$. In the case $p = 1$, partial results are known but computational issues remain open.

[2] There is a subtle difference here between dimension 2 and higher. Indeed, a function holomorphic on a plane domain is defined by its non-tangential limit on a boundary subset of positive linear measure, but there are non-constant harmonic functions in the 3-D ball, C^1 up to the boundary sphere, yet having vanishing gradient on a subset of positive measure of the sphere. Such a "bad" subset, however, cannot have interior points on the sphere.

[3] Though harmonic function theories on half-spaces and balls are equivalent through Kelvin transforms, conformal maps are severely restricted when $n > 2$, so that general domains Ω can no longer be normalized; related Hardy spaces have not been much studied so far.

Just like solving problem (P) appeals to the solution of a standard extremal problem when $K = \mathbb{T}$, our ability to solve problem (P_n) will depend on the possibility to tackle the special case where $\Gamma = \mathbb{S}$. This is a simple problem when $p = 2$ by virtue of the Hardy-Hodge decomposition together with orthogonality of $H^2(\mathbb{B})$ and $H^2(\mathbb{R}^n \setminus \overline{\mathbb{B}})$, which is the reason why we were able to solve (P_n) in this case. Other values of p cannot be treated as easily and are still under investigation, especially the case $p = \infty$ which is of particular interest and presents itself as a 3-D analog to the Nehari problem [74]. A companion to this problem is the one below, where the space of tangent divergence-free fields on $\mathbb{S}$ is denoted by $D(\mathbb{S})$.

> Let $1 \leq p \leq \infty$ and $F \in L^p(\mathbb{S})$ be a $\mathbb{R}^n$-valued vector field. Find $G \in H^p(\mathbb{B})$ and $D \in D(\mathbb{S})$ such that $\|G + D - F\|_{L^p(\mathbb{S})}$ is minimum.

This question is especially relevant to electro-encephalography (EEG) and inverse magnetization issues, see Sections 4.1 and 4.2. The latter problem can be reduced to the former in 2-D, since divergence-free vector fields on $\mathbb{R}^2$ supported on $\mathbb{T}$ are real multiples of $ie^{i\theta}$, but it is no longer so in higher dimension. Both problems arise in connection with inverse potential problems in divergence form, see Sections 3.2.2, 3.2.3.

Conductivity equation. Similar approaches can be considered for more general equations than the Laplacian, for instance isotropic conductivity equations of the form $\mathcal{L}\phi = \mathrm{div}(\sigma\nabla\phi) = 0$ where σ is no longer a constant function but admits positive values[4]. Then, the relevant Hardy spaces in Problem (P) are those associated to a so-called conjugate Beltrami equation: $\bar{\partial} f = \nu\overline{\partial f}$ [62], with $\nu = (1-\sigma)/(1+\sigma)$, which are studied for $1 < p < \infty$ in [2, 32, 42, 50]. Expansions of solutions needed to constructively handle such issues in the specific case of linear fractional conductivities have been expounded in [56]. Studying Hardy spaces of conjugate Beltrami equations is of interest in its own right. For Sobolev-smooth coefficients of exponent greater than 2, they were investigated in [32, 42]. The case of the critical exponent 2 is treated in [2], which provides an initial example of well-posed Dirichlet problem in the non-strictly elliptic case: the conductivity may be unbounded or zero on sets of zero capacity and, accordingly, solutions need not be locally bounded. More importantly perhaps, the exponent 2 is also the key to a corresponding theory on general rectifiable domains in the plane, as coefficients of pseudo-holomorphic functions obtained by conformal transformation onto a disk are no better than L^2-integrable in general, even if the initial problem has higher summability. Such generalizations are under study, in collaboration with Élodie Pozzi (Saint Louis University, Missouri, USA) and Emmanuel Russ (Aix-Marseille Université), and nontrivial connections between the regularity of the conformal parameterization of the domain and the range of exponents p for which the Dirichlet problem is solvable in L^p have been brought to light. In fact, for Lipschitz domains at least, this range of exponents coincides with the interval of p for which the modulus of the derivative of a conformal map from the unit disk onto Ω satisfies the so-called A_p property of Hunt-Muckenhoupt-Wheeden.

Such generalized Hardy classes were also used in [28] where to address the uniqueness issue in the classical Robin inverse problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with uniformly bounded Robin coefficient, L^2 Neumann data and conductivity of Sobolev class $W^{1,r}(\Omega)$, $r > n$. We showed that uniqueness of the Robin coefficient on a subset of the boundary, given Cauchy data on the complementary part, does hold in dimension $n = 2$, thanks to a unique continuation result, but needs not hold in higher dimension. This raises an open issue on harmonic gradients for $n \geq 3$, namely whether positivity of the Robin coefficient is compatible with identical vanishing of the boundary gradient on a subset of positive measure.

3.1.2 Data extension problems

Closely related to the inverse problems are problems of the data extension type. They differ from Cauchy-type problems of Section 3.1.1 in that the given data are located in the interior of the domain Ω of validity of the equation rather than on its boundary $\partial\Omega$. More precisely, these problems arise when the solution of an elliptic PDE with unknown coefficients or boundary data is given in some sub-region of Ω, and one wishes to extend this knowledge to a bigger region there. Determining the solution in the bigger region may be the primary goal, but motivation to consider such extension problems may also come from an attempt at improving solution to an inverse problem (*e.g.*, an inverse source estimation problem, see Section 3.2.3), or generating additional information making that problem amenable to asymptotic methods where an asymptotic parameter is related to the size of the data region (see Section 3.4).

[4]Note that for constant conductivities σ, we are back to the above case of the Laplacian, and that for smooth enough σ, the conductivity PDE can be reformulated as a stationary Schrödinger equation.

While extension problems are generally easier than inverse problems since one may avoid the non-uniqueness issue, usually the extension process is still unstable and appropriate regularization must be used as long as data are not exact.

Due to the high regularity of solutions to elliptic PDEs away from the support of the source term, many extension problems can be addressed using certain types of analytic continuation.

A relevant example to the class of applied problems considered by Factas (see Section 4.2) is given by the Poisson equation $\Delta\phi = \operatorname{div} m$ (where $\mathcal{L}\phi = \Delta\phi$ and $\Psi(m) = \operatorname{div} m$) in $\mathbb{R}^3$ with an unknown square summable $\mathbb{R}^3$-valued function m supported in $S \times \{0\}$ for some bounded set $S \subset \mathbb{R}^2$. It is assumed that ϕ is measured on $S \times \{h\} \subset \Omega$ for some $h > 0$, Ω being the upper half-space. The primary goal is to estimate ϕ on $\widetilde{S} \times \left\{\widetilde{h}\right\}$ for some set $\widetilde{S}$, such that $S \subset \widetilde{S} \subseteq \mathbb{R}^2$ and $\widetilde{h} \geq h$. This issue can be solved through an extension problem which seeks to determine the scalar valued function ϕ on $\mathbb{R}^2 \times \{h\}$. The final goal is then achieved (if $\widetilde{h} > h$; otherwise this step is not even needed) by the so-called upward continuation process, *i.e.*, computing the Poisson transform of ϕ on $\mathbb{R}^2 \times \{h\}$ with the "height" parameter $\widetilde{h} - h$. Note that, by formulating such an extension problem, we have by-passed the full inverse source problem of finding the function m, that admits non-unique solutions. An illustration of such an approach for data extension in context of the inverse magnetization problem can be found, for example, in [75].

3.1.3 Spectral issues

Solving inverse problems by a linear least-square approach leads to an equation featuring the operator $A^\star A$, with A being the forward operator governing the direct problem, mapping m to the measurements (potential ϕ or field $\nabla\phi$), and $A^\star$ its adjoint (in appropriate function spaces). The range of the operator A is usually strictly smaller than the model space for measurements, hence the solution of the inverse problem is unstable. When A is a compact integral operator, stability analysis and regularization can be achieved through singular-value decomposition. When additionally A is self-adjoint and of convolution type (convolution operators naturally arise as inverses of differential ones), the situation reduces to the eigenvalue problem for a convolution operator, typically on a bounded region. Here, not only the study of rates of convergence of eigenvalues to zero is important (to quantify and mitigate the ill-posedness of the original inverse problem), but also the determination of the corresponding eigenfunctions as they lead to efficient practical implementations. Finding an explicit form of eigenfunctions is a virtually impossible task, for solutions of convolution integral equations on a finite region are explicitly solvable only on extremely rare occasions, even in a one-dimensional setting. Consequently, one naturally resorts to numerical methods and asymptotic constructions when the region is large. However, the latter are still challenging for integral equations with kernel functions related to Poisson kernel. This motivated the Factas members' work [41] which, together with its further generalization [76], extend explicit asymptotic constructions beyond the classes of integral equations where previous results were applicable [61, 64, 67]. Finally, we note that since convolution operators are closely related to Toeplitz operators, this makes contact with numerical inversion of large Toeplitz matrices [48].

3.2 Inverse source problems

Given an elliptic PDE of the form $\mathcal{L}\phi = \Psi(m)$ as in Section 3.1, the corresponding inverse source problem consists in recovering the quantity m from the data, that typically consist of measurements of the potential ϕ or the field away from the support of the source. Usually the support of m is assumed to be compact, and a super-set thereof is specified. This kind of issues may thus be seen as parameterized inverse potential problems (parameterized by m, that is). They arise naturally in non-destructive testing, medical imaging, paleo-magnetism, gravimetry and geosciences, as well as in inverse scattering, see Section 4. The second and third application domains pertain to static electromagnetism, a framework in which source terms typically occur in divergence form; that is, $\Psi(m) = \operatorname{div} m$ where m is a $\mathbb{R}^3$-valued field or distribution. As a rule, the operator $\mathcal{L}$ is then of the form $\mathcal{L}\phi = \operatorname{div}(\Sigma\nabla\phi)$, where Σ is valued in positive definite matrices satisfying fixed ellipticity bounds and relates to the electromagnetic characteristics of the medium. In this case, the problem amounts to recover a vector field (namely: $m/\det(\Sigma)$) knowing a super-set S of its support and the gradient summand of its Helmholtz decomposition outside of S, for the Riemannian metric with tensor $(\det\Sigma)\Sigma^{-1}$. The simplest case, of course, occurs in the Euclidean setting, where $\mathcal{L}$ is the ordinary Laplacian, which corresponds to homogeneous media.

3.2.1 Hardy-Hodge decomposition

In its original form, the Hardy-Hodge decomposition allows one to express a $\mathbb{R}^n$-valued vector field in $L^p(\mathbb{S}^{n-1})$, $1 < p < \infty$, as the sum of a vector field in $H^p(\mathbb{B}^n)$, a vector field in $H^p(\mathbb{R}^n \setminus \overline{\mathbb{B}^n})$, and a tangential divergence free vector field on $\mathbb{S}^{n-1}$. Here, $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ are, respectively, the open unit ball of $\mathbb{R}^n$ and its boundary sphere, while $H^p(\mathbb{B}^n)$ (resp. $H^p(\mathbb{R}^n \setminus \overline{\mathbb{B}^n})$) are the classical harmonic Hardy spaces of Stein-Weiss; namely, gradients of harmonic functions in $\mathbb{B}^n$ (resp. $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$) whose L^p-norm over spheres centered at 0 are uniformly bounded, identified with their nontangential limit on $\mathbb{S}^{n-1}$ which allows one to consider them as $\mathbb{R}^n$-valued vector fields of L^p-class *on the sphere*. If $p = 1$ or $p = \infty$ the decomposition fails, but a natural substitute is that a $\mathbb{R}^n$-valued vector field with components not just in $L^1(\mathbb{S}^{n-1})$ but in the real Hardy space $H^1(\mathbb{S}^{n-1})$ does have a Hardy-Hodge decomposition, whose summands have components in $H^1(\mathbb{S}^{n-1})$; and a vector field whose components are in $L^\infty(\mathbb{S}^{n-1})$ has a Hardy-Hodge decomposition whose components lie in $BMO(\mathbb{S}^{n-1})$, the space of functions with bounded mean oscillation on $\mathbb{S}^{n-1}$. The decomposition is in fact valid more generally on any C^1-smooth surface, and even on Lipschitz surfaces if one restricts the range of p to an interval around 2 [43]. It appears to play a fundamental role in inverse potential problems, and was first introduced on the plane to describe silent magnetizations supported in $\mathbb{R}^2$ [36] (see Sections 3.2.2 and 4.2). It has been a forerunner to similar decompositions where Hardy spaces in a domain get replaced by silent sources in that domain [34], and there are currently attempts at generalizing it to more general elliptic operators than the Laplacian. In fact, the Hardy-Hodge decomposition can be viewed as a Hodge decomposition in degree 1 of currents of L^p-class supported on a hyper-surface in ambient Euclidean space, and generalizing the former to more general elliptic operators amounts to generalize the latter to more general Riemannian metrics.

3.2.2 Silent sources

A salient feature of inverse source problems is that the forward operator A is often not injective. The nature of its null-space depends both on the function Ψ making $\Psi(m)$ the source, on the geometry of the set S containing the support of m and the smoothness of the model class, as well as on the type of measurements. Abusing terminology slightly, those m belonging to that null space are called “silent sources” even though, strictly speaking, the source term is $\Psi(m)$ rather than m.

The occurrence of nontrivial silent sources hinders most approaches to inverse source problems, and their study appears to be necessary in order to derive consistent regularization schemes. Indeed, discretizing beforehand will typically turn an inverse problem with non-injective forward operator A into a full rank but ill-conditioned finite-dimensional one, whereas the very structure of the null space could yield an ansatz that may restore uniqueness, for example normalized representatives or suitable notions of sparsity for m, which in turn suggest appropriate regularization terms when minimizing the distance between the outcome of the model and the data.

This point of view leads one to state and approximately solve continuous optimization problems depending on some chosen regularization method, and is similar in spirit to an “off-the-grid” approach as in [55]. The fact that inverse source problems for elliptic PDE can be recast in terms of integral forward operators, using Green functions, only adds to the comparison with the reference just mentioned. However, a major difference with the approach developed there is that the so-called “source condition” will almost never hold in our case, which prevents analogous consistency estimates to apply.

When the source term is in divergence form; *i.e.*, when $\Psi(m) = \operatorname{div} m$, and if we assume that the measurements are faithful, there are roughly speaking two possibilities: either S has Lebesgue measure zero and does not separate the space, in which case silent sources are divergence free ($\operatorname{div} m = 0$), or S fails to meet one of these conditions and the class of silent sources becomes considerably larger. In the former case one says that S is slender, and the distinction between the slender and non-slender cases is apparent from the works [34, 36, 46].

Silent sources in the slender case can be described rather completely when m is modeled by $\mathbb{R}^3$-valued functions or measures. Notions of sparsity have been drawn from this characterization [36], and several types of constructive approaches to reconstruction and net moment estimation, with different assumptions and algorithms, are currently under study by Factas. In contrast, silent sources in the non-slender case were understood only recently for L^p-functions on domains which are not too wild, see [12, 35] together with the PhD thesis [72] of Masimba Nemaire. Reconstruction algorithms in this case are still in their infancy.

Besides, understanding silent sources when m is a vector measure is tantamount to characterize when the Helmholtz-Hodge decomposition of such a measure again consists of measures. This is an open issue in harmonic analysis.

Silent sources of L^2-class on a closed Lipschitz surface were also analyzed in the Factas team for the Helmholtz equation with (possibly complex) wave number k, namely $\Delta\phi+k^2\phi = \mathrm{div} m$ (where $\mathcal{L}\phi = \Delta\phi+k^2\phi$, $\Psi(m) = \mathrm{div} m$), in collaboration with H. Haddar from the Idefix project team, with applications to the modeling of non-isotropic scattering. The situation is a bit more involved than with the Laplacian (that is: when $k = 0$) and depends on whether k is a Neumann eigenvalue or not [11].

3.2.3 Source estimation

A classical approach to inverse problems is to minimize with respect to the unknown m, belonging to a model class E_1 (usually a Banach space), a criterion of the form $d(f, A\,m) + F(\lambda, \|m\|)$, where $A : E_1 \to E_2$ is the forward operator, assumed to be compact and mapping E_1 into a measurement space E_2 endowed with a metric d, while f is the data and $F(\lambda, \|m\|)$ is a smooth positive penalty term, depending in an increasing manner on a non-negative regularizing parameter λ, that satisfies $\partial_x F(0, x) > 0$ for all $x > 0$. The identification scheme then consists in estimating the unknown m_0 by a minimizer of the criterion, for some appropriate small positive value of λ designed to offset the measurement error involved with f (in standard applications, A has dense range). A minimal requirement, then, is that this identification scheme should be consistent in the limit, when the measurement error e goes to zero and the regularization parameter λ also goes to zero, in a manner that may depend on e. When the forward operator is injective, such consistency is more or less automatic, at least in a weak sense; but when it is not injective, consistency can only be achieved upon making additional assumptions on m_0, that ensure it is a solution of minimum norm to $f = Am_0$. This is why the penalty term F should be chosen in relation to the null space of A.

Let us now specialize to inverse source problems for the Laplacian with right hand side in divergence form: $\Delta\phi = \mathrm{div} m$ (*i.e.*, $\mathcal{L}\phi = \Delta\phi$, $\Psi(m) = \mathrm{div} m$), assuming that the forward operator A is faithful, compact and, say, valued in a Hilbert space $\mathcal{H}$ (it could be a measurement of the field in a region of space away from the source). A common, yet simplifying hypothesis in EEG source problems (Section 4.1), is to assume that the support of m is contained in a closed surface S homeomorphic to a ball (an idealized model of gray matter), in such a way that m is normal to S and of L^2-class there. Then, standard properties of layer potentials imply that m is uniquely determined by the field, so the forward operator is injective in this case and consistency is guaranteed (under the previous hypothesis). This example contrasts the next one, which is important for inverse magnetization problems (Section 4.2): assume that m ranges over $\mathbb{R}^3$-valued measures supported on a set of zero Lebesgue measure that does not disconnect the Euclidean space $\mathbb{R}^3$ (the so-called slender case). Then, the null space of A consists of divergence free measures; see Section 3.2.2. Now, by a result of S. Smirnov [81] such measures are superpositions of line integrals, therefore measures whose support contains no arc (a so-called “purely 1-unrectifiable set”) are mutually singular to the null space of A. Consequently, for $\mathbb{R}^3$-valued measures whose support is sparse in that it contains no arc, consistent identification schemes can be obtained upon minimizing $\|f - Am\|^2_{\mathcal{H}} + \lambda\|m\|_{TV}$ with respect to m, with $\|\cdot\|_{TV}$ to indicate the total variation norm. This was expounded in [47] and ongoing research recently showed that the same holds for more general elliptic operators than the Laplacian. This is an example of how the null space of the forward operator can suggest an ansatz on the solution (here a measure-theoretic notion of sparsity) and impinge on the choice of the regularization.

The non-slender case, that involves important frameworks for S like closed surfaces or volumes, is less understood. Deriving interesting ansatz for $\mathbb{R}^3$-valued measures in connection with the kernel of the forward operator in such situations is subject to ongoing research within Factas.

Of course, this approach to inverse source problems requires to solve infinite-dimensional optimization problems, which in turn calls for some discretization techniques. A classical idea, pervading throughout numerical analysis, is to approximate the solution of such an infinite-dimensional problem by a sequence of solutions to finite-dimensional ones. In the slender case, a suitable sequence of finite-dimensional optimization problems can be obtained by replacing the space of $\mathbb{R}^3$-valued measures supported on S by a k-dimensional subspace $\mathcal{M}_k$ thereof, and arrange things so that a nested sequence $\mathcal{M}_k \subset \mathcal{M}_{k+1} \subset \cdots$ is weakly dense in the space of such measures. Then, one is left to solve a sequence of finite-dimensional least square problems with l_1 constraints. Convergence issues are currently being addressed within Factas in collaboration with colleagues at Vanderbilt University and the University of Vienna. The absence of a source condition makes such developments a novel piece of research.

When the space E_1 where the unknown m is sought is a Hilbert space, replacing a test space of functions by a sequence of nested finite-dimensional sub-spaces, as outlined above in the case of $\mathbb{R}^3$-valued measures, is also suggestive of L-curve methods from singular value decomposition to approximately solve infinite-dimensional linear equations like $Am = f$, for A a compact operator. A possible regularization parameter is then the number of terms retained in the singular vectors expansion, and the main difficulty is, of course, to numerically estimate sufficiently large number of such singular vectors in a precise enough manner (see Sections 3.1.3 and 8.1).

We also mention that solving less ambitious inverse problems than source reconstruction is often regarded as a more attainable, but still valuable endeavor. In particular, for inverse magnetization problems (see Sections 4.2 and 8.2), this can be said of net moment recovery. Unlike the magnetization m, its net moment, the integral of m, is simply a vector which is entirely determined by the field, because silent sources have zero moment. Hence, it should be considerably easier to estimate. Nevertheless, this task is far from trivial in practice, mainly because field measurements are performed in a limited region of space and are thus incomplete. The design of net moment estimators is another avenue explored by Factas, after initial work in this direction reported in [31, 38] and, more recently, in [79].

3.3 Rational approximation, behavior of poles

Rational approximation to holomorphic functions of one complex variable is a long standing chapter of classical analysis, with notable applications to number theory, spectral theory and numerical analysis. Over the last decades, it has become a cornerstone of modeling in Systems Engineering, and it can also be construed as a technique to regularize inverse source problems in the plane, where the degree is the regularizing parameter. Indeed, by partial fraction expansion, a rational function can be viewed as the complex derivative of a discrete logarithmic potential with as many masses as the degree (assuming that the poles are simple); that is, if f is rational of degree N, then $\bar{\partial} f = \sum_{j=1}^{N} a_j \delta_{z_j}$ where the z_j are its poles and δ_{z_j} is a Dirac unit mass at z_j. Moreover, a holomorphic function is the complex derivative of the logarithmic potential of its own values on a curve encompassing the domain of analyticity (this is the Cauchy formula); hence, rational approximation aims at representing as well as possible a logarithmic potential by a discrete potential with prescribed number of masses, in the sense that their derivatives should be close (a Sobolev-type approximation).

Predecessors of Factas (the Apics and Miaou project teams) have designed a dedicated steepest-descent algorithm for quadratic approximation criteria whose convergence to a local minimum is guaranteed. This gradient algorithm may either be initialized by a preliminary approximation method, or recursively proceed with respect to the degree N of the approximant, on a compactification of the parameter space [29], as can be done with the RARL2 software (see Section 7.1.3). It has proved to be effective in applications carried out by the team (see for instance [9] for the identification of micro-wave filters, and Sections 4.1 and 4.3).

However, finding best rational approximants of prescribed degree to a specific function, say in the uniform norm on a given set, seems out of reach except in rare, particular cases. Instead, constructive rational approximation has focused on estimating optimal convergence rates and deriving approximation schemes coming close to meet them, or studying computationally appealing approximants like Padé interpolants and their variants. Two main issues are then the effective computation of optimal or near optimal approximants of given degree, and the connection between the singularities of the approximant (the poles) and those of the approximated function. Factas has contributed to both.

We studied in particular the behavior of best rational approximants of given degree, in the L^∞-sense on a compact subset of the domain of analyticity, to complex analytic functions f that can be continued analytically (possibly in a multi-valued manner) except perhaps over a set of logarithmic capacity zero in the plane. When the continuation of f has finitely many branches; that is, if the Riemann surface to which this continuation extends analytically in a single-valued manner (except perhaps on a polar subset thereof) is compact, then the behavior as N goes large of rational approximants of degree N whose N-th root error is asymptotically smallest possible (in particular the asymptotic behavior of best approximants) has recently been elucidated rather completely in this joint research effort involving Factas.

As regards near-optimal approximants, their design requires a knowledge of optimal rates in the situation at hand. In recent years, we were active determining lower bounds on that rate, a piece of information which is crucial but difficult to obtain. Our methods are topological in nature (Ljustenik-Schnirelman theory, genus of compact symmetric sets), like most techniques in the area, and in collaboration with Tao Qian from the

University of Macao, we devised algorithms to compute lower bounds in best L^2 approximation by stable rational function of given degree on the unit circle, which is first of this kind and sometimes quite precise, see [5]. Research in this direction is still active, in particular on best L^2 approximation of functions of constant modulus, which is an old issue in system theory (how to perform model reduction of delay systems) that has received a new lease of life from the heavy trend of neural networks. Such functions cannot be approximated in uniform norm, except when they are rational of admissible degree, in which case they are, of course, their own best approximant. Their L^2 rational approximation is possible, though, but not very efficient and the problem is to quantify this fact by giving a lower bound on the achievable approximation error by a rational function of given degree.

Another classical technique to approximate –more accurately: extrapolate– a function, given a set of pointwise values, is to compute a rational interpolant of minimal degree to match the values. This method, known as Padé (or multi-point Padé) approximation has been intensively studied for decades [27] but fails to produce pointwise convergence, even if the data are analytic. The best it can give in general, at least to functions whose singular set has capacity zero, is convergence in capacity which does not prevent poles of the approximant from wandering about the domain of analyticity of the approximated function, but does imply that each pole of the approximated function attracts a pole of the approximant [69]. This phenomenon is well-known in numerical analysis, and leads Physicists and Engineers to distinguish between "mathematical" and "physical" poles. A modification of the multi-point Padé technique, in which the degree is kept much smaller than the number of data and approximate interpolation is performed in the least-square sense, has become especially popular over the last decade under the name vector fitting; it teams up with a barycentric representation of rational functions satisfying prescribed interpolation conditions, known as AAA (for Anderson-Antoulas Adaptive) scheme. Though the behavior of this least square substitute to Padé approximation, defined by Equation (1) in Section 4.4, resembles the one of multi-point Padé approximants from a numerical viewpoint, there has been apparently no convergence result for such approximate interpolants so far. Motivated by the outcome of numerical schemes developed by our partners to recover resonance frequencies of conductors under electromagnetic inverse scattering, the PhD thesis [25] of Paul Asensio started investigating the behavior of such least-square rational approximants to functions with polar singular set, and dwelling on this work, we were recently able to show convergence in capacity thereof.

Regarding complex rational approximation as a means to tackle inverse source problems in the plane makes for a unifying point of view on various deconvolution techniques, from system identification and time series analysis to frequency-wise inverse scattering and non-destructive testing. But still more interestingly perhaps, it is suggestive of similar approaches to problems in higher dimension, where holomorphic functions generalize to harmonic gradients and rational functions to finite linear combinations of dipoles, see Section 3.1.2. This line of research is only starting, but seems to offer new avenues in connection with applications.

3.4 Asymptotic analysis

Asymptotic analysis deals with understanding behavior or explicit construction of the solution when a parameter entering a problem is either small or large. Factas has been involved in applications of asymptotic analysis in different contexts including both formal constructions and their rigorous justifications.

One type of asymptotic analysis for dynamical problems is the large-time behavior analysis. A rather classical issue here is that of limiting amplitude principle for wave equation. This principle states that the solution of the time-dependent wave equation with a periodic-in-time monochromatic source term $fe^{i\omega t}$ necessarily stabilizes for large times t to the solution of the corresponding Helmholtz equation: $\mathrm{div}(\alpha\nabla\phi) + \omega^2\beta\phi = -f$ for suitable known functions α, β and f depending on space variables. Revisiting this topic is motivated by recent popularity of numerical time-domain approaches (see, *e.g.*, [22, 23, 60, 83]) to elliptic PDE problems through efficient solution of auxiliary time-dependent equations where, for example, computational effort needed to be concentrated only on wave front neighborhood which is small for high-frequencies, a notoriously difficult regime for numerical solution of Helmholtz problems. With this respect, not only the fact of the time convergence in the limiting amplitude principle is important but also quantification of its rate. Dmitry Ponomarev was involved in the work [1] that deals with the limiting amplitude principle for a non-homogeneous medium wave-equation in different dimensions (in the dimension 1, a slight modification of the classical limiting amplitude principle was proposed). The analysis approach relies on proving that the solution decays to zero for a source-free problem with localized initial data and radiation boundary condition, an interesting problem of asymptotic analysis on its own, even in dimension 1 [24]. In [78], convergence to the periodic motion was also shown in a totally different context of one model of mechanical sliding contact problem with wear [77].

Previously, in a nonlinear context, rigorous asymptotic analysis [73] was instrumental to justify a parabolic model of pulse propagation in photo-polymers by comparing solutions of that model with those of the original Maxwell's system.

In the context of inverse problems, asymptotic analysis is useful when applied to the magnitude of the regularization parameter. When the latter tends to its limiting value, a solution of the regularized problem with ideal (noiseless) input data should tend to the exact solution. In presence of noise, it is important to relate this convergence rate to the value of the problem's constraint in the asymptotic regime of the regularization parameter.

The works [41, 76] on convolution integral equations on large domains, mentioned in Section 3.1.3, are an example of constructive asymptotic analysis. Here, one of the difficulty comes from a singular perturbation. Indeed, in the asymptotic limit of infinite size of the region, the spectral problem solution cease to exist since the integral operator loses the compactness property.

In the context of net magnetization reconstruction in the inverse magnetization problem (Sections 3.2 and 4.2), situation when the measurement area size is large leads to a different kind of application of constructive asymptotic analysis [31, 38, 79]. Here, explicit constructions of the solution estimates are performed to different asymptotic orders with respect to the measurement region size. The higher-order estimates can give good accuracy already for relatively small value of the measurement region but are much more unstable with respect to the perturbation of the measured data. This problem also exhibits another interesting asymptotic phenomenon which is somewhat similar to the "boundary layer" common for boundary-value problem for differential equations with small or large parameters. In particular, the solution (for tangential components of net moment) is composed of a global leading-order quantity (where formal passage to the asymptotic limit can be performed) and a correction term which is localized in a region that shrinks in the asymptotic limit.

4 Application domains

Most of the targeted applications by the team pertain to the context of Maxwell's equations, under various specific assumptions.

4.1 Some inverse problems for cerebral imaging

Solving over-determined Cauchy problems for the Laplace equation on a spherical layer (in 3-D) in order to extrapolate incomplete data (see Section 3.1.1) is an ingredient of the team's approach to inverse source problems in electro-encephalography (EEG), see [51]. The model comes from Maxwell's equation in the quasi-static regime, whence $\operatorname{div}(\sigma\nabla\phi) = \operatorname{div} m$ in a ball Ω which is the union of nested layers Ω_i (brain, skull, scalp for $i = 0, 1, 2$), where the singularities lie in Ω_0 (*i.e.*, the current sources m are supported in $S \subset \Omega_0$, with the notation of Section 3), see Figure[5] 1. The inverse EEG source problem consists in recovering m from pointwise values of the electric potential ϕ measured on the scalp Γ_2 (together with the assumption that the normal current vanishes there). It first involves transmitting the data from the scalp Γ_2 down to the cortex Γ_0. Whenever the Ω_i are of different constant conductivities, ϕ satisfies Laplace equation in the outermost shells Ω_2 and Ω_1 and this "cortical mapping" step is performed using integral representations of ϕ on the spheres Γ_i (obtained through convolution by the Poisson kernels of the balls and using Green formula, a specific use of boundary element methods), followed by expansions on spherical harmonic bases and Tikhonov regularization.

Assuming m to be a linear combination of Dirac masses (dipolar sources), it turns out (by convolution with the fundamental solution of 3-D Laplace equation) that traces of ϕ on 2-D cross sections of Γ_0 coincide with functions with branched singularities in the slicing plane [39, 45]. These singularities are related to the actual location of the sources. Hence we are back to the 2-D framework of Section 3.3, and recovering these singularities can be performed via best rational approximation. The goal is to produce a fast and sufficiently

[5]Observe that Figure 1 actually describes a setup related to SEEG, see below, and distributed source terms m, in a more general geometrical setting.

accurate initial guess on the number and location of the sources in order to run heavier descent algorithms on the direct problem, which are more precise but computationally costly and often fail to converge if not properly initialized. Such a localization process can add a geometric, valuable piece of information to the standard temporal analysis of EEG signal records. It appears that, in the rational approximation step, multiple poles possess a nice behavior with respect to branched singularities. This is due to the very physical assumptions on the model from dipolar current sources, for both EEG data and MEG (magneto-encephalography) data that correspond to measurements of the magnetic field, as well as for (magnetic) field data produced by magnetic dipolar sources within rocks (see Section 4.2). Though numerically observed in [51], there is no mathematical justification so far why branched singularities generate such strong accumulation of the poles of the approximants. This intriguing property, however, is definitely helping source recovery and will be the topic of further study. It is used in order to automatically estimate the "most plausible" number of sources (numerically: up to 3, at the moment). In this connection, a software FindSources3D (FS3D, see Section 7.1.4) dedicated to pointwise source estimation in EEG–MEG has been developed. We also studied the uniqueness of a critical point of the quadratic criterion in the EEG source problem in Ω_0 for a single dipole situation (see [25]), an important issue for the use of descent algorithms.

Together with Marion Darbas (LAGA, Université Sorbonne Paris Nord) and Pierre-Henri Tournier (JLL laboratory, Sorbonne Université), we recently considered the EEG inverse problem with a variable conductivity in the intermediate skull layer Ω_1, in order to model hard / spongy bones, especially for neonates. The related transmission step is then performed using a mixed variational regularization and finite elements on tetrahedral meshes, and the coupling with FS3D for dipolar source estimation furnishes promising results [8].

Other approaches have been studied for EEG, MEG and "Stereo" EEG (SEEG), where the potential is measured by deep electrodes and sensors within the brain as in the scheme of Figure 1, and for more realistic geometries of the head. Assuming that the current source term m is a $\mathbb{R}^3$-valued vector field (measure, or

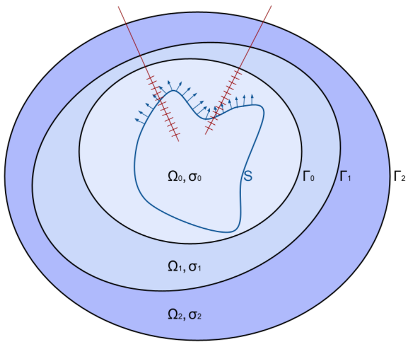

Figure 1: Schematic view of a 3 layered head model. Deep SEEG electrodes with sensors along them (red) in Ω_0, current source term m distributed on $S \subset \Omega_0$ with normally oriented dipoles.

distribution) supported on a surface $S \subset \Omega_0$ (the gray / white matters interface within the brain) and normally oriented to S, they consist in regularizing the inverse source problem by a total variation (TV) constraint - to favor sparsity - on m, added to the quadratic data approximation criterion (see Section 3.2). This is similar to the path that is taken for inverse magnetization problems (see Section 4.2). The approach follows that of [7] and is implemented through algorithms whose convergence properties were studied in [25, 72]. We are now able to handle MEG, EEG, SEEG modalities, simultaneously or not. The simultaneous handling of the different modalities is made more straightforward by coupling the source localization problem and the inverse transmission problem. Tests on synthetic data provided good quality results, though they are quite numerically costly to obtain. This opens up the possibility to consider sources that may exhibit properties usually associated with distributions rather than functions.

4.2 Inverse magnetization issues for planar and volumetric samples

Among other things, geoscientists are interested in understanding the magnetic characteristics of ancient rocks. Indeed, ferro-magnetic particles in a rock carry a magnetization that has been acquired when the rock was hot, under the influence of the magnetic field that was ambient at that time. For an igneous rock for instance, and if no subsequent event has heated it up, this corresponds to the time when the rock was formed. If the rock can be dated, recovering its magnetization hence provides valuable information about the history of the magnetic field. This gives elements for better understanding to key questions such as: what was the magnetic field of the sun when the solar system was at the proto-planetary phase? when did the magnetic dynamo of Mars stop? when did the magnetic dynamo of Earth start?

The magnetization of a rock is not directly measurable. However, it produces a tiny magnetic field, which can be measured if the sample is isolated from other sources of magnetic field. A category of instruments of particular importance with that respect are the magnetic microscopes. They are used to measure the field produced by a fairly small sample: either a simple grain, or a wider sample that has been first prepared by gluing it on some support and polishing it until getting only a thin slab. The microscope operates at some distance above the sample and measures the magnetic field. The typical experimental set up is represented on Figure 2.

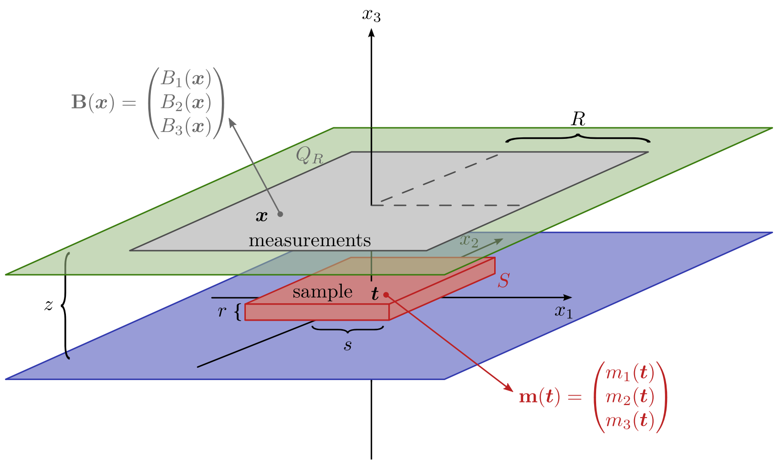

Figure 2: Schematic view of the experimental setup of a magnetic microscope. The sample lies on a horizontal plane at height 0 and its support is included in a parallelepiped (in red). The field produced by the sample is measured at points of a horizontal region, say a square, at height z (in gray).

The magnetization m is modeled as a $\mathbb{R}^3$-valued vector field defined on the rock sample which is assumed to be a subset of S. One advantage of the magnetic microscopes is that they operate close to the sample, *i.e.*, the height z of measurement is small compared to the horizontal characteristic lengths s and R. The thickness of the sample is also small compared to s. However, the ratio r/z is not necessarily negligible. In the cases when it is indeed negligible, the sample can be supposed planar instead of being volumetric (*i.e.*, $r = 0$) from a practical point of view; in this case, t becomes a planar variable $(t_1, t_2) \in S$ (a square) and m is actually a magnetization density.

The surface Q_R of measurements is, most of the time, supposed to be a centered square of half-size R, but in some situations it might be convenient to consider only the data available on a centered disk D_R of radius R. The microscope provides a map of the field on the whole surface: measurements are provided at many points of the surface and, from a practical point of view, one may assume that the field is known everywhere on Q_R. In addition to the presence of noise in the measurements, an important limitation is that, depending on the microscope technology, it is frequent that only one component of the field be measured.

The team has a long-standing collaboration with the Earth and Planetary Sciences Laboratory at MIT. They have a superconducting quantum interference device (SQUID). The sensor is a tiny vertical coil maintained at temperature close to 0 Kelvin, which provides it with superconducting characteristics. In order to maintain the sensor at very low temperature while the microscope operates in a room at normal temperature, the sensor is isolated behind a sapphire window. Though thin, this window enforces a measurement height z such that $r \ll z \ll s$ and the sample can usually be supposed planar. Also, because the coil only allows to measure the field along its axis, the SQUID only provides measurements of B_3 and not the whole field.

Another type of microscopes consists in the quantum diamond microscopes (QDM). They use properties of special diamonds which, when properly excited with a laser and a microwave field, become luminescent in the presence of a magnetic field. From the difference of brightness of this luminescence under slightly different frequencies of the microwave field, one can recover the amplitude of a given component of the magnetic field. This mechanism is already in use to provide a magnetic microscope at Harvard University (Massachusetts, USA). We are collaborating with geoscientists of the Geophysics and Planetology Department of Cerege (CNRS, Aix-en-Provence) and physicists from ENS Paris Saclay to help them designing their own QDM. The promises of the QDM are manifolds, see also Section 8.2. First, they should allow measurements of the field at a height z above the sample that is way smaller than what is permitted with the SQUID. This improves the spatial resolution and together with the conditioning of the inverse magnetization problem. However, this also imposes to model the sample as a 3-D object, as its thickness r becomes usually comparable to z in this case. Second, the technology of the QDM could make it possible, in principle, to measure the three components of the field instead of only one, which opens the way to new regularization techniques for the inverse problems. However, the measurement of a field with a QDM is more indirect than with a SQUID and this raises specific issues: in particular, it might be necessary to impose an external field on the sample, called a bias field, in order to resolve ambiguities when recovering the field from the brightness of the luminescence, and such a bias field can actually perturb the magnetic properties of the rock sample under study.

The issues raised by the inverse magnetization problem in the framework of magnetic microscopes such as SQUIDs or QDMs are numerous and we got several contributions on the subject over time. Particularly important for the full recovery of the magnetization are the silent sources, *i.e.*, the magnetization that belong to the kernel of the direct operator, or in other terms, those magnetization that produce no field on the measurement area, see Section 3.2.2. We fully characterized such magnetizations in the thin-plate hypothesis (*i.e.*, when r is assumed to be 0), [36]. Contrary to the full magnetization, the total net moment (*i.e.*, the integral of the magnetization over the sample) is in principle a piece of information that could be retrieved from the measurements, since silent sources have a null net moment. In order to recover the total net moment, we proposed to use linear estimators and defined a bounded extremal problem to find good such linear estimators [4]. However, additional hypotheses must be added in order to ensure the uniqueness of a solution for the full inversion problem. Such an hypothesis is provided when the support of the magnetization inside the sample is supposed to be sparse, *e.g.*, composed of isolated points or 1-D curves. In this case, we proposed to use total variation regularization to solve the inverse problem [7, 47], see Section 3.2.3.

4.3 Inverse magnetization issues with the lunometer

Measurements of the remanent magnetic field of the Moon let geoscientists think that the Moon used to have a magnetic dynamo for some time, but the exact process that triggered and fed this dynamo is not yet understood, much less why it stopped. In particular, the Moon is too small to have a convecting dynamo like the Earth has. In order to address this question, our geoscientists colleagues at Cerege decided to systematically analyze the rock samples brought back from the Moon by Apollo missions.

The samples are kept inside bags with a protective atmosphere, and geophysicists are not allowed to open the bags, nor to take out the samples from NASA facilities. Moreover, the measurements must be performed with a passive device in order to ensure that the samples would not be altered by the measurements: in particular no cooling or heating is allowed, and neither is the use of anything producing a magnetic field like, *e.g.*, motors. Finally, since the measurements must be performed directly at NASA, the instrument must be easy to take apart and to assemble once on site. The overall time devoted to measuring all samples is limited and each sample must be analyzed quickly (typically within a few minutes). For all these reasons, our colleagues from Cerege designed a specific magnetometer called the “lunometer”: this device provides measurements of the components of the magnetic field produced by the sample, at some discrete set of points located on disks belonging to three cylinders (see Figure 3). The goal was not to get a deep understanding of the magnetic properties of the studied samples with such a rudimentary instrument but rather to help selecting a few of them that seems really interesting to study in more details: this would be used to file a request to NASA to buy sub-samples of a few grams on which more instructive (though possibly destructive) experiments could be performed.

The collaboration with Cerege on this topic started in the framework of the MagLune ANR project whose overall objective was to devise models to explain how a dynamo phenomenon was possible on the Moon. Our contribution is to design methods to tell, from the measurements provided by the Lunometer, whether the remanent magnetization of the sample under study could be well modeled by a single magnetic dipole, and if so, what would be the position and magnetic moment of this dipole. To this end, we use ideas similar

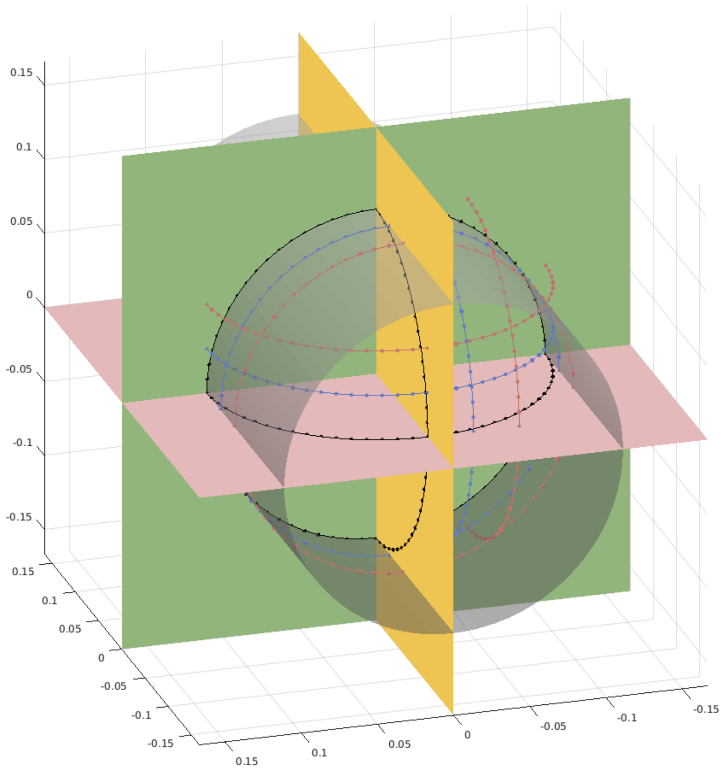

Figure 3: Typical measurements obtained with the lunometer of Cerege. Measurements of the field are performed on nine circles, given as sections of three cylinders. On each circle, only one component of the field is measured: the component B_h along the axis of the corresponding cylinder (blue points), the component B_n radial with respect to the circle (black points), or the component B_τ tangential to the circle (red points).

to those underlying the FindSources3D tool (see Sections 3.3 and 7.1.4): we use rational approximation techniques to recover the position of the dipole; recovering the moment is then a rather simple linear problem. The rational approximation solver gives, for each circle of measurements, a partial information about the position of the dipole. These partial informations obtained on all nine circles must then be combined in order to recover the exact position. Theoretically speaking, the nine partial informations are redundant and the position could be obtained by several equivalent techniques. But in practice, due to the fact that the field is not truly generated by a single dipole, and also because of noise in the measurements and numerical errors in the rational approximation step, all methods do not show the same reliability when combining the partial results. We studied several approaches, testing them on synthetic examples, with more or less noise, in order to propose a good heuristic for the reconstruction of the position [70].

4.4 Shape identification of metallic objects

We started an academic collaboration with partners at LEAT (Laboratoire d'Électronique, Antennes, Télécommunications, Université Côte d'Azur – CNRS) on the topic of inverse scattering using frequency dependent measurements. As opposed to classical electromagnetic imaging where several spatially located sensors are used to identify the shape of an object by means of scattering data at a single frequency, a discrimination process between different metallic objects is here being sought for by means of a single, or a reduced number of sensors that operate on a whole frequency band. The spatial multiplicity and complexity of antenna sensors is here traded against a simpler architecture performing a frequency sweep.

The setting is shown on Figure 4. The total field E_t is the sum of the incident field E_{in} (here a plane wave) and scattered field E_s: at every point $X = (r, \theta, \varphi)$ in space we have $E_t = E_{in} + E_s$. A harmonic time dependency ($e^{i\omega t}$), is supposed for the incident wave, so that by linearity of Maxwell equations and after a transient state, the scattered field at the observation point X_o is related to the emitted planar wave field at the emission point X_e via the transfer function H: $E_s(X_o) = H(\omega, X_o)\, E_{in}(X_e)$ (the dependency in X_e is omitted in H since the emission point is fixed). Under regularity conditions on the scatterer's boundary, the function H can be shown to admit an analytic continuation into the complex left half-plane for the s variable, away from a discrete set (with a possible accumulation point at infinity) where it admits poles. Thus, H is a meromorphic function in the variable s. Its poles are called the resonating frequencies (resonances) of the scattering object. Recovering these resonating frequencies from frequency scattering measurement, that is measurements of H at particular $\omega_j' s$ (actually, $i\,\omega_j' s$) is the primary objective of this project. We started a

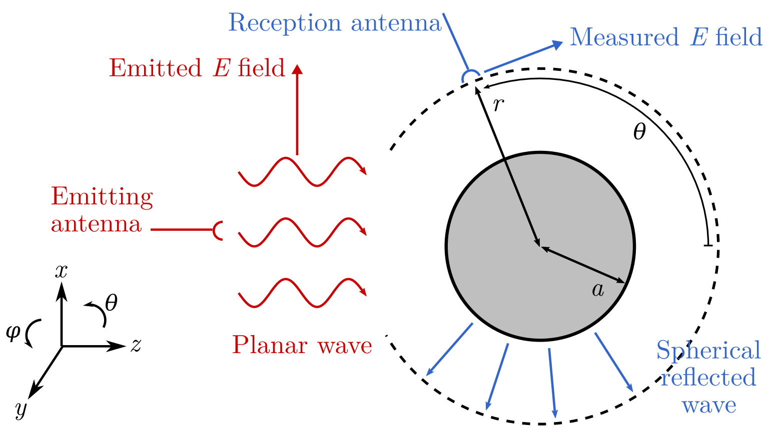

Figure 4: Sphere illuminated by an electromagnetic plane wave - measurement of the scattered wave.

study of the particular case when the scatterer is a spherical PEC (Perfectly Electric Conductor). In this case, Maxwell equations can be solved by means of expansions in series of vectorial spherical harmonics. We showed in particular that in this case H admits a simple structure involving a meromorphic function with poles at zeroes of the spherical Hankel functions and their derivatives. Identification procedures, surprisingly close to the ones we developed in connection with amplifier stability analysis (Section 4.6), were studied to gain information about the resonating frequencies by means of a rational approximation of this function [86].

In order to perform the rational approximation (see Section 3.3), the behavior of H outside the range of measured frequencies, specifically at high frequencies, has been studied for the particular case when the scatterer is a spherical PEC (Perfectly Electric Conductor). In this case, H can be written as $H = H_O + H_C$, where H_O and H_C are respectively the optic and creeping wave parts. Their high-frequency behaviors are given by series expansions whose coefficients are identified when $X_o = X_e$. The asymptotics of H_O is called the Luneberg-Kline expansion; its first terms were analytically computed in [25] (solving eikonal and transport equations).

Numerical simulations showed that even though H_C is negligible with respect to H_O at high frequencies, it needs to be taken into account around the band of measured frequencies for the rational approximation. Furthermore, the physical interpretation of these two terms leads to consider that H_C should carry more information about the scatterer and we want to investigate the conjecture that the poles of H are those of H_C hence that H_O is analytic.

The rational approximation of the transfer function H is performed with a least-squares substitute to multi-point Padé approximation: the approximant $R_{k,n}[H] = p_{k,n}/q_{k,n}$ is given by:

$$(p_{k,n}, q_{k,n}) = \operatorname{argmin}_{p\in\mathcal{P}_n,\, q\in\mathcal{P}_k^1} \sum_{j=1}^{N} \left|p(z_j) - H(z_j)q(z_j)\right|^2, \tag{1}$$

where n, k and N are integers such that $k + n + 1 \leq N$ and $(z_j)_{j=1\ldots N}$ is a collection of points at which H is analytic. Here, $\mathcal{P}_n$ is the set of polynomials of degree less than n and $\mathcal{P}_k^1$ is the set of monic polynomials of degree k. An analog of the Nuttal-Pommerenke theorem for a least square version of classical Padé approximants was obtained in [25], where the values $(p - H\,q)(z_j)$ in Equation (1) get replaced by the j-th coefficients of the Taylor expansion of $(p - H\,q)$ at a given point in the domain of analyticity of H. It says that if H is holomorphic and single-valued on $\mathbb{C}$, except perhaps on a polar set, then p/q converges to H in capacity as k, n go to infinity as well as N, in such a way that n/k remains bounded and $N \leq C(k + n)$ for some constant $C > 0$; convergence in capacity means that for each $\varepsilon > 0$, the capacity of the set where the pointwise error is bigger than ε goes to zero.

Dwelling on this work, we recently showed under similar conditions on k, n, N, and provided that the interpolation points z_i remain in a bounded set, that convergence in capacity still holds for the solutions to Equation (1); this is a least square analog of Wallin's theorem in multi-point Padé approximation.

This result is interesting as it entails that poles of H can indeed be retrieved as limit points of certain poles of $p_{k,n}/q_{k,n}$, while explaining the chaotic behavior of other, so-called spurious poles that wander about the domain of analyticity. We plan to investigate other PEC scatterers.

4.5 Inverse problems in orthopedic surgery

Apart from more classical medical imaging domains, inverse problems find a rather surprising application in the field of orthopedics, see [58, 68, 71] and Section 9.2.

We are concerned, in particular, with a hip prosthetic surgery when an insertion of an acetabular cup (AC) implant into a bone by press-fit is performed with the use of an instrumented hammer. Such a hammer is equipped with a sensor capable to measure impact momentum (force) and hence yield important information about the bone quality and the stability of an AC implant. These are, indeed, crucial pieces of information to have in real-time during a surgery. On the one hand, if the achieved bone-implant contact area is not sufficiently large, osseointegration may fail eventually leading to an aseptic loosening of the implant and a necessity of a revision surgery. On the other hand, if the insertion of the implant is too deep, the generated stresses may induce fractures or bone tissue necrosis.

The mathematical side of the process is far from trivial. First of all, contact mechanics is a highly nonlinear problem due to geometrical constraint on the solution. Already a basic problem of an elastic body on a rigid foundation is a free-boundary problem with an unknown effective contact surface which is characterized by the so-called Signorini conditions, nonlinear constraints of Karush-Kuhn-Tucker type involving stress and displacements. In the present case, several coupled problems have to be solved since regions corresponding to the bone, the implant and the hammer all possess different material properties. Moreover, the deformations cannot be considered small, consequently, a hypo-elastic description is more appropriate than that of linear elasticity. In such a formulation, a rate relation between Cauchy stress and strain tensors replaces a linear stress-strain constitutive law, hence its integration induces additional non-linearity.

Finally, the bone is a porous multi-scale medium, and appropriate homogenization model should be deduced, with adequate parameter fitting.

For tackling inverse problems (*e.g.*, that of determining material parameters), the direct formulation has to be solved in such an effective way that iterative approaches are not prohibitively expensive. This motivates exploration of model-order reduction strategies that would, in particular, allow efficient integration of the system in time.

4.6 Stability and design of active devices

Through contacts with CNES (Toulouse) and UPV (Bilbao), the team got involved in the design of amplifiers which, unlike filters, are active devices. A prominent issue here is stability. Twenty years ago, it was not possible to simulate unstable responses, and only after building a device could one detect instability. The advent of so-called harmonic balance techniques, which compute steady state responses of linear elements in the frequency domain and look for a periodic state in the time domain of a network connecting these linear elements via static non-linearities made it possible to compute the harmonic response of a (possibly nonlinear and unstable) device [84]. This has had tremendous impact on design, and there is a growing demand for software analyzers. In this connection, there are two types of stability involved. The first is stability of a fixed point around which the linearized transfer function accounts for small signal amplification. The second is stability of a limit cycle which is reached when the input signal is no longer small and truly nonlinear amplification is attained (*e.g.*, because of saturation).

Initial applications by the team have been concerned with the first type of stability, and emphasis was put on defining and extracting the "unstable part" of the response. We showed that under realistic dissipativity assumptions at high frequency for the building blocks of the circuit, the linearized transfer functions are meromorphic in the complex frequency variable s, with at most finitely many unstable poles in the right half-plane [3]. Dwelling on the unstable/stable decomposition in Hardy Spaces, we developed a procedure to assess the stability or instability of the transfer functions at hand, from their evaluation on a finite frequency grid [54], that was further improved in [53] to address the design of oscillators. This has resulted in the development of a software library called Pisa[6], aiming at making these techniques available to practitioners. Extending this methodology to the strong signal case, where linearization is considered around a periodic trajectory, is considerably more difficult and has received much attention by the team in recent years. The exponential stability of the high frequency limit of a circuit was established in [6], implying that there are at most finitely many unstable poles and no other unstable singularity for the monodromy operator around the cycle. Furthermore, the links between the monodromy operator and the (operator-valued) "harmonic transfer function" (HTF) of the linearized system along the trajectory were brought to light in [10]: the system is exponentially stable if and only if its HTF is bounded an analytic in a right half-plane of $\mathbb{C}$ of the form $\{\Re z > -\varepsilon\}$ for some $\varepsilon > 0$. We deal here with input-output system of the form:

$$y(t) = \sum_{j=1}^{N} D_j(t)\, y(t-\tau_j) + u(t), \qquad t > t_0$$

where $\tau_1 < \cdots < \tau_N$ are positive delays and $D_1(t), \ldots, D_N(t)$ real $d \times d$ matrices depending periodically on time t, while u is the $\mathbb{R}^d$-valued input and $y(t)$ the $\mathbb{R}^d$-valued output (complex coefficients can of course be handled in the same way). The HTF, that generalizes the usual transfer function of linear constant systems, is an analytic function of the Laplace-Fourier variable which is valued in the space of operators on $L^2([0,T),\mathbb{C}^d)$ (T is the period of the system). It can be defined as follows: if a periodic linear control system at rest is fed from initial instant $t_i = -\infty$ with an input signal $v(t)\, e^{xt} e^{i\nu t}$ where v is T-periodic and $x > 0$ is large enough, then the output is of the form $\mathbf{H}(x+i\nu)(v)e^{xt}e^{i\nu t}$, where $\mathbf{H}(x+i\nu)$ is the harmonic transfer evaluated at $x+i\nu$ (an operator) and $\mathbf{H}(x+i\nu)(v)$ is the T-periodic function which is the image of the T-periodic function v under $\mathbf{H}(x+i\nu)$. That is to say, an exponentially modulated input wave of frequency ν carried by a T-periodic signal is mapped to an output of the same form, and the HTF maps the input carrier to the output carrier. For stable systems, this is asymptotically true if the initial time is a finite instant; otherwise, the unstable transient will prevent this mathematical solution from being physical. Other definitions are given in [10].

[6]See the website for details.

We were able to construct a simple nonlinear circuit whose linearization around a periodic trajectory has a spectrum containing a whole circle; we currently investigate whether the singularities of the Fourier coefficients of the HTF (that are themselves analytic functions) also contain that circle. Indeed, just like a series of functions may diverge even though the summands are smooth, it is a priori possible that the HTF has a singularity at a point whereas its Fourier coefficients do not. Note that these coefficients are all one can estimate by harmonic balance techniques, and therefore the above question is of great practical relevance. We also investigate the relation between our stability criterion (that the HTF should be bounded and analytic on a "vertical" half-plane containing the origin), and the weaker requirement that the HTF exists pointwise in a half-plane. Connections with the representation of Volterra equations with jumps in the kernel are also a motivation for such a study, see [33].

4.7 Tools for numerically guaranteed computations

The overall and long-term goal is to enhance the quality of numerical computations. This includes developing algorithms whose convergence is proved not only when assuming that the numerical computations are performed in exact real or complex arithmetic, but rather when really accounting for the fact that the computations are performed with an inexact arithmetic (usually floating-point arithmetic). A numerical result alone is of little interest if no rigorous bound is provided together with it, in order to ensure that the real theoretical result is proved to be not to far from the computed result.

A specific way of contributing to this objective is to develop efficient numerical implementations of mathematical functions with rigorous bounds. We do sometimes provide such implementations. The software tool Sollya (see Section 7.1.1), developed together with Christoph Lauter (University of Texas at El Paso, UTEP) is also an achievement of the team in that respect. This tool intends to provide an interactive environment for performing numerically rigorous computations. Sollya comes as a standalone tool and also as a C library that allows one to benefit from all the features of the tool in C programs.

5 Social and environmental responsibility

- Sylvain Chevillard and Martine Olivi are members of the organizing committee of the RESET seminar (Redirection Écologique et Sociale : Échanges Transdisciplinaires), an inter-lab and interdisciplinary seminar in Sophia, dedicated to the themes of ecological transition and sustainable development.

- Sylvain Chevillard co-animated a "Ma Terre en 180 minutes" workshop (a three-hours workshop where participants participate to a role play of a research laboratory committed into dividing by two its carbon footprint, and looking for practical solutions to reach this goal) with two other people for the staff of Université Côte d'Azur on the Valrose campus.
He is further involved in teaching environmental issues at Polytech Sophia Antipolis (see Section 10.2).

- Martine Olivi was a member of the CLDD (Commission Locale de Développement Durable).
She is a member of Labos1point5, an international, cross-disciplinary collective of academic researchers who share a common goal: to better understand and reduce the environmental impact of research, especially on the Earth's climate, and a member of the GdRS EcoInfo (CNRS).
She participated in the creation of the exhibition Exposition pour la Sobriété Numérique dans l'ESR and gave a talk on digital sufficiency (what's stopping us from getting started?) at "Numerique en commun 2025 Alpes-Maritimes" (slides available online).

6 Highlights of the year

6.1 Awards

Mubasharah Khalid Omer was awarded a poster prize at the 5th edition of Complex Days, Nice (February), see Section 10.1.4 and [21].

6.2 Working conditions

One of our PhD students got serious health problems in 2024, that led to a temporary interruption of his progress. As he was being back to work beginning of 2025, after recovering thanks to medical care, he has been subject to a decision[7] from the prefecture (Alpes-Maritimes) to leave the French territory ("obligation de quitter le territoire français, OQTF"), hence forced back to Morocco (his birth country). Granted such poor conditions, it turned out that the completion of his PhD was impossible, despite the fact that we all had hoped for a more favorable end of his stay.

7 Latest software developments, platforms, open data

7.1 Latest software developments

7.1.1 Sollya

Keywords: Floating-point, Remez algorithm, Supremum norm, Multiple-Precision, Interval arithmetic

Functional Description: Sollya is an interactive tool where the developers of mathematical floating-point libraries (libm) can experiment before actually developing code. The environment is safe with respect to floating-point errors, i.e. the user precisely knows when rounding errors or approximation errors happen, and rigorous bounds are always provided for these errors.

Among other features, it offers a fast Remez algorithm for computing polynomial approximations of real functions and also an algorithm for finding good polynomial approximants with floating-point coefficients to any real function. As well, it provides algorithms for the certification of numerical codes, such as Taylor Models, interval arithmetic or certified supremum norms.

It is available as a free software under the CeCILL-C license.

News of the Year: This year, we started to implement a new command called fpapproxl2, which is a companion to the fpminimax command. Like fpminimax, fpapproxl2 finds a good approximation to a function by a polynomial whose coefficients fit on given floating-point or fixed-point formats. And like fpminimax it does it by reducing the problem to finding a vector in a Euclidean lattice that is as close as possible to a given point in the underlying space. However, unlike fpminimax, it is fully based on the L2 norm and does not need to internally run (or be feeded with the result of) the remez command. Indeed, the core algorithm is now provided by another new command, called fpapprox, which takes an argument that can be 2 or infty and that determines which variant to use. Most of the implementation is done, but documentation and tests are still to be written before the new features be officially included in the tool. This is a joint work with Tom Hubrecht from Pascaline Inria team in Lyon.

URL: https://sollya.org/

Publication: hal-00761644

Contact: Sylvain Chevillard

Participants: Christoph Lauter, Tom Hubrecht, Jérôme Benoit, Marc Mezzarobba, Mioara Joldes, Nicolas Jourdan, Sylvain Chevillard

7.1.2 PRESTO-HF

Keywords: CAO, Telecommunications, Microwave filter

Functional Description: Presto-HF is a toolbox dedicated to low-pass parameter identification for microwave filters. In order to allow the industrial transfer of our methods, a Matlab-based toolbox has been developed, dedicated to the problem of identification of low-pass microwave filter parameters. It allows one to run the following algorithmic steps, either individually or in a single stroke:

[7] The student filed an appeal with the administrative court. The decision was eventually judged abusive and cancelled end of 2025.

• Determination of delay components caused by the access devices (automatic reference plane adjustment),

• Automatic determination of an analytic completion, bounded in modulus for each channel,

• Rational approximation of fixed McMillan degree,

• Determination of a constrained realization.

For the matrix-valued rational approximation step, Presto-HF relies on RARL2. Constrained realizations are computed using the Dedale-HF software. As a toolbox, Presto-HF has a modular structure, which allows one for example to include some building blocks in an already existing software.

The delay compensation algorithm is based on the following assumption: far off the pass-band, one can reasonably expect a good approximation of the rational components of S11 and S22 by the first few terms of their Taylor expansion at infinity, a small degree polynomial in 1/s. Using this idea, a sequence of quadratic convex optimization problems are solved, in order to obtain appropriate compensations. In order to check the previous assumption, one has to measure the filter on a larger band, typically three times the pass band.

This toolbox has been licensed to (and is currently used by) Thales Alenia Space in Toulouse and Madrid, Thales airborne systems and Flextronics (two licenses). XLIM (University of Limoges) is a heavy user of Presto-HF among the academic filtering community and some free license agreements have been granted to the microwave department of the University of Erlangen (Germany) and the Royal Military College (Kingston, Canada).

News of the Year: In 2025, we changed the license of Presto-HF and made it open source under the GPL license. Besides getting the legal agreements, this required to create a public git repository, to clean up the files in order to include only the ones essential to make Presto-HF work and to include headers with copyrights and disclaimer notices in every files.

URL: https://project.inria.fr/presto-hf/

Contact: Fabien Seyfert

Participants: Fabien Seyfert, Jean-Paul Marmorat, Martine Olivi

7.1.3 RARL2

Name: Réalisation interne et Approximation Rationnelle L2

Keyword: Approximation

Functional Description: RARL2 is a software for rational approximation. It computes a stable rational L2-approximation of specified order to a given L2-stable (L2 on the unit circle, analytic in the complement of the unit disk) matrix-valued function. This can be the transfer function of a multivariable discrete-time stable system. RARL2 takes as input either:

• its internal realization,

• its first N Fourier coefficients,

• discretized (uniformly distributed) values on the circle. In this case, a least-square criterion is used instead of the L2 norm.

It thus performs model reduction in the first or the second case, and leans on frequency data identification in the third. For band-limited frequency data, it could be necessary to infer the behavior of the system outside the bandwidth before performing rational approximation.

An appropriate Möbius transformation allows to use the software for continuous-time systems as well.

The method is a steepest-descent algorithm. A parametrization of MIMO systems is used, which ensures that the stability constraint on the approximant is met. The implementation, in Matlab, is based on state-space representations.

RARL2 is distributed under a particular license, allowing unlimited usage for academic research purposes. It was released to the universities of Delft and Maastricht (the Netherlands), Cork (Ireland), Brussels (Belgium), Macao (China) and BITS-Pilani Hyderabad Campus (India).

News of the Year: In 2025, we decided to open the source code of RARL2 and distribute it under the GPL license. We are still waiting for the legal agreements of all involved institutions to effectively do so. In the perspective of this publication of the code, we started gathering historical revisions of the RARL2 software, in order to prepare a clean view of the development history to be included in a public git repository. We also fixed a few bugs.

URL: http://www-sop.inria.fr/apics/RARL2/rarl2.html

Contact: Martine Olivi

Participants: Jean-Paul Marmorat, Martine Olivi

7.1.4 FindSources3D

Keywords: Health, Neuroimaging, Visualization, Compilers, Medical, Image, Processing

Functional Description: FindSources3D (FS3D) is a software program written in Matlab dedicated to the resolution of inverse source problems in brain imaging, EEG and MEG. From data consisting in pointwise measurements of the electrical potential taken by electrodes on the scalp (EEG), or of a component of the magnetic field taken on a helmet (MEG), FS3D estimates pointwise dipolar current sources within the brain in a spherical layered model (when simultaneously available, EEG and MEG data can be processed together, which improves the recovery performance).

In the situation of 3 spherical head layers, the electrical conductivities of the innermost and outermost layers (brain and scalp) are assumed to be constant, while the conductivity of the intermediate (skull) one could be variable. The time dependency is either neglected and the data processed instant by instant, or separated from the space behavior using a singular value decomposition (SVD). Next, a transmission step ("cortical mapping") from the scalp to the brain's boundary is performed, followed by a best rational approximation step of traces of the transmitted potential on families of 2D planar cross-sections. From the obtained collection of poles, the 3D sources are finally estimated in a last clustering step.

Through this process, FS3D is able to recover time correlated sources, which is an important advantage with respect to other software tools addressing the same problem.

URL: http://www-sop.inria.fr/apics/FindSources3D/en/index.html

Publication: hal-03880526v2

Contact: Juliette Leblond

Participants: Jean-Paul Marmorat, Juliette Leblond, 3 anonymous participants

8 New results

8.1 Field extrapolation for inverse magnetization problem and beyond

Participants: Axel Knecht, Juliette Leblond, Mubasharah Khalid Omer, Dmitry Ponomarev.

Motivation of the field extrapolation problem has been described in the last year's activity report of the team.

One of the approaches, based on regularized deconvolution (see Section 4.2), is the topic of the PhD thesis work of Mubasharah Khalid Omer. Over the last year not only different basis functions were explored, but also discretization, various ways of choosing the regularization parameter and alternative projection and iterative formulations for the solution of the normal equation. Further, numerical tests validated the applicability of this approach for volumetric magnetizations.

A second method, the so-called double-spectral vector approach [75], which preserves the magnetization localization, has been revised, with its altered version explored (when the role of the kernel functions entering it was swapped). Moreover, alternative scalar spectral approaches have been proposed based on a similar idea of individual continuation of eigenfunctions of appropriate integral operators. Comparison has been made and the paper outlining extrapolation methods based on such spectral methods is currently in preparation.

Furthermore, we investigated a possibility of taking advantage of the auxiliary quantities that were used in each case to generate the extrapolated field. Namely, it became clear that, in the case of planar source support (or very thin sample), both the divergence of the tangential magnetization and the vertical component of magnetization can indeed be reconstructed, either directly (as for the double-spectral approach) or by means of solving an additional ill-posed problem (as with the deconvolution approach). The latter indicates that the extrapolation approach can indeed be considered as a flexible first step of the full inverse magnetization problem. The aforementioned auxiliary quantities can also be used to directly calculate the magnetization net moment components, which provides an alternative bypassing the use of the asymptotic formulas with the extrapolated field.

We note that the knowledge of the tangential (two-dimensional) divergence of the planar magnetization allows the reconstruction of the magnetization itself modulo silent sources whose structure is well-understood (see Sections 3.2.2 and 4.2 and [30]). Two constructive strategies for recovering the unique magnetization of minimal quadratic norm equivalent to the given one (*i.e.* with the same tangential divergence and the normal component) have been investigated in the internship work of Axel Knecht. In particular, one of these strategies was shown to be especially computationally efficient due to the local formulation of the problem (avoiding repetitive evaluations of auxiliary integral operators over the entire plane).

8.2 Net moment estimation

Participants: Sylvain Chevillard, Juliette Leblond, Jean-Paul Marmorat, Dmitry Ponomarev, Fatima Swaydan, Anass Yousfi.

We continued the work started in the past years to establish formulas for the integrals of the form

$$\iint_{\text{domain}} P(\boldsymbol{x}) B_j(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{2}$$

where P is a polynomial, $j \in \{1, 2, 3\}$, and the domain of integration is either the square Q_R (and more generally, any rectangle) or the disk D_R of radius R (the notations are those of Section 4.2, see especially Figure 2). While we were historically focusing on the case $j = 3$ because it corresponds to the measurements obtained with the SQUID, we now try to get as much results as possible to be valid for any component $j \in \{1, 2, 3\}$, in view of the measurements that could be soon obtained with the QDM.

In the case of the disk, Equation (2) does not admit an exact explicit expression. However Dmitry Ponomarev described in [80], a few years ago, asymptotic expansions of moderate order with respect to variable R as it grows large in the case when $j = 3$ and P has degree less than 2. On the one hand, these results have been extended to higher orders and higher degrees and published this year in [13]. On the other hand, Anass Yousfi contributed a considerably simpler proof of these formulas during his PhD, while extending the results to any component B_j. They are available in the research report [18].

Even though a technique for obtaining estimates of arbitrary high order was devised, it was also observed in [13] that such high-order estimates are more sensitive to measurement noise and thus might be of a little value in practice. Mid-order estimates were numerically shown to be the best in a situation mimicking the realistic set-up where the data are corrupted by noise and are not available over a large region. Remarkable features of the derived asymptotic estimates are that they are applicable for fairly general magnetization distributions (say, a combination of dipoles) and that their application does not require the knowledge of the distance from the sample to the measurement plane.

The case when the domain is a rectangle allows for more detailed results than just asymptotic estimates. More specifically, denoting by abuse of notations B_j the (forward) operator from $L^2(S)$ to $L^2(Q_R)$ that maps a magnetization to the corresponding component of the field ($j \in \{1, 2, 3\}$), its adjoint operator $B_j^\star$ plays an important role for the estimation of the net moment from measured data since the quality of a linear estimator is good when its image by the adjoint operator is close to be a characteristic function. This year, we completely wrote down computations that were only sketched in a conference paper in 2024: we showed that the adjoint operator admits nice integral representations and, pushing forward similar computations done by Jung for small degrees [63], we derived exact and explicit expressions for the image of any polynomial by the adjoint operator [17]. These expressions can be used either to straightforwardly get asymptotic formulas of the same kind as those described above in the disk geometry, or more generally, can be handy in any context when the evaluation of the adjoint operator is important, such as, *e.g.*, to numerically determine the solution of the extremal problem presented in [4].

We also started to take advantage of the 3-component field measurements provided by QDM for the reconstruction of net moments through the solution of an auxiliary bounded extremal problem, see [4]. This is the topic of the PhD thesis of Fatima Swaydan. Some preliminary results are reported in [20], where a linear combination of 3 components of the magnetic field is considered as data, and a numerical study of the spectral properties of the corresponding operator was performed. The follow-up considerations will concern further exploration of such linear combinations for net moment estimation, depending also on the considered constraints or regularization term and parameters, in the underlying bounded extremal problem.

8.3 Inverse conductivity estimation problems in cerebral imaging

Participants: Juliette Leblond, Jean-Paul Marmorat, Rui Martins, Dmitry Ponomarev.

For the spherical and piecewise constant conductivity head model (see Section 4.1), we now consider together with Marion Darbas (LAGA, Université Sorbonne Paris Nord) the issue of simultaneous recovery of dipolar source terms and of the intermediate (skull) conductivity value from partial Dirichlet-Neumann data on the outermost boundary (EEG setup), following [51, 52]. We were able to establish a uniqueness result and an iterative alternate estimation procedure. Preliminary numerical computations from synthetic data using the software FS3D (see Section 7.1.4) are encouraging. Stability, robustness, and convergence properties, remain under study.

A quite new imaging modality is MDEIT (Magnetic Detection Electrical Impedance Tomography), where upon injection of an electrical current pattern through surface electrodes, the response of the media is probed through a measurement of the produced magnetic field (instead of measuring the electrical potential). Preliminary simulations predict better results than the conventional electrical impedance tomography (EIT) approach. Moreover, it was already shown that measuring all three components of the magnetic field leads to much better conditioning and thus much faster convergence of iterative schemes for the conductivity reconstruction. This was the topic of the internship of Rui Martins and remains an on-going work with him and colleagues at University College London (UCL).

8.4 General Inverse Source Problems in Divergence Form

Participants: Laurent Baratchart.

Inverse source problems with source term in divergence form consist, roughly speaking, in finding a vector field with prescribed support whose divergence is the Laplacian of a potential, given some measurements of that potential (see Section 3.2). Here the Laplacian could be either the usual Euclidean one or a more general, elliptic operator in divergence form arising as the Laplacian of some Riemannian metric. From the point of view of geometric analysis, such questions amount to retrieving the divergence free term in the Helmholtz decomposition of a vector field, knowing the gradient term on some observation set Q and a superset S of the support of the unknown vector field. Such problems appear in various contexts including Geomagnetism, Paleomagnetism, as well as Medical Imaging from EEG and Electro-Cardiography (ECG) and are severely ill-posed with non-unique solutions, making regularization techniques an essential aspect of every approach.

If the unknown vector field is denoted by m, the potential $\phi = \phi(m)$ satisfies $\mathrm{div}\,(\Sigma\nabla\phi) = \mathrm{div}\, m$ on $\mathbb{R}^3$ with Σ an elliptic symmetric matrix-valued function, and it vanishes at infinity. When m is assumed to be a $\mathbb{R}^3$-valued compactly supported measure (a standard model in electro/magnetostatics as it includes dipoles), this equation has a wild right hand side and is not standard. We showed this year that it has a unique very weak solution vanishing at infinity, provided Σ is uniformly elliptic and Dini-continuous in a neighborhood of the support of m; more generally, we proved such well-posedness when Σ is merely uniformly elliptic, in some even weaker sense where test functions depend on m. Under slightly stronger assumptions, namely when Σ is Dini-continuous in a neighborhood of S and Q, then the regularized minimization problem

$$\inf_{m, \mathrm{supp} m \subset S} \|\nabla\phi(m) - f\|_{L^2(Q)} + \lambda\|m\|_{TV}$$

has a unique solution for fixed $\lambda > 0$, with f arbitrary data in $L^2(Q)$. This result supersedes a uniqueness result dealing with the Euclidean Laplacian and planar samples obtained in [46].

8.5 Rational and meromorphic approximation

Participants: Laurent Baratchart, Sylvain Chevillard.

Rational approximation on the circle or the line to functions of constant modulus is an old issue; in the language of system identification and control, it is akin to ask how well stationary delay systems can be represented by finite-dimensional ones, and it has long been surmised that such a representation should be inefficient for all reasonable input-output criteria. In recent years, approximation in L^2-norm, which can be interpreted as minimum variance approximation in a probabilistic context, received renewed interest due to the advent of (linear) neural nets. Still, it seems that no quantitative lower bound for the approximation error is known in terms of the respective degrees of the approximated function and its approximant (see Section 3.3). Through a collaborative research effort with Alexander Borichev, Claire Coiffard and Rachid Zarouf (Aix-Marseille Université), triggered by a question of Alexandre François (Sierra team, centre Inria de Paris), we established such a bound by combining a formula expressing the degree in terms of the $W^{1/2,2}$-norm due to Brézis and new estimates of the Fourier coefficients of Blaschke products.

Let us also mention that an article is still under writing on convergence in capacity of least square substitutes to multi-point Padé approximants (1) to functions with polar singular set on $\mathbb{C}$, a topic that underwent advances last year after groundbreaking work in [25].

9 Partnerships and cooperations

9.1 International research visitors

9.1.1 Visits of international scientists

Other international visits to the team

Richard Huber

Status post-doc

Institution of origin: Technical University of Denmark (DTU)

Country: Denmark

Dates: May, 4-11

Context of the visit: preparation for CRCN/ISFP applications

Mobility program/type of mobility: lecture

Eduardo A. Lima

Status researcher

Institution of origin: MIT

Country: USA

Dates: October, 5-11

Context of the visit: joint research, preparation for MIT-France proposal

Mobility program/type of mobility: research stay

Rui C. A. Martins

Status PhD

Institution of origin: University of Aveiro

Country: Portugal

Dates: September-December

Context of the visit: joint research related to MDEIT

Mobility program/type of mobility: internship, research stay

9.2 National initiatives

ANR R2D2 ANR-21-IDES-0004, "Welcome package" (2022–2027) attributed to Dmitry Ponomarev.

Participants: Dmitry Ponomarev, Fatima Swaydan.

It supports the PhD thesis of Fatima Swaydan, see Section 8.2.

ANR MoDyBe ANR-23-CE45-0011-03, "Modeling the dynamic behavior of implants used in total hip arthroplasty" (2023–2028). Led by the laboratory Modélisation et Simulation Multi Échelle, UMR CNRS 8208 and the Université Paris-Est Créteil (UPEC), involving Factas team, together with the department of Orthopedic surgery of the Institut Mondor de Recherche Biomédicale, U955 Inserm and UPEC.

Participants: Juliette Leblond, Dmitry Ponomarev.

Cement-less implants are increasingly used in clinical practice of arthroplasty. They are inserted in the host bone using impacts performed with an orthopedic hammer (press-fit procedure, see Section 4.5). However, the rate of revision surgery is still high, which is a public health issue of major importance. The press-fit phenomenon occurring at implant insertion induces bio-mechanical effects in the bone tissues, which should ensure the stability of the implant during the surgery ("primary stability"). Despite a routine clinical use, implant failures, which may have dramatic consequences, still occur and are difficult to anticipate. Just after surgery, the implant fixation relies on the pre-stressed state of bone tissue around the implant. In order to avoid aseptic loosening, a compromise must be found by the surgeon. On the one hand, sufficient primary stability can be ensured by minimizing micro-motion at the bone-implant interface in order to promote osteo-integration phenomena. On the other hand, excessive stresses in bone tissue around the implant must be avoided, as they may lead to bone necrosis or fractures. This raises the following mathematical issues. What is the appropriate mechanical model of the implant insertion process into the bone? What are the suitable high-performance computing methods to accurately solve the above modeling equations for the bone-implant interaction subject to dynamic excitations? Which robust inversion approaches can be employed to retrieve the quantities of interest of the bone-implant interaction such as the bone-implant contact area? The ANR project MoDyBe aims to address these issues. It will support the PhD thesis of Gouda Chérif Bio, starting January 2026.

Réseau Thématique.

Participants: Laurent Baratchart, Sylvain Chevillard, Juliette Leblond, Martine Olivi, Dmitry Ponomarev.

Factas is part of the "réseau thématique" *ANAlyse et InteractionS* (ANAIS). It gathers people doing fundamental and applied research concerning function spaces and operators, dynamical systems, auto-similarity, probabilities, signal and image processing.

10 Dissemination

10.1 Promoting scientific activities

10.1.1 Scientific events: organization

General chair, scientific chair

- Laurent Baratchart was the organizer of a mini-symposium at Inverse Problems, Control, and Shape Optimization (PICOF), Hammamet, Tunisia (October).
- Laurent Baratchart and Dmitry Ponomarev organized a mini-symposium at the Shanks conference, Nashville, Tennessee, USA (May).

Member of the organizing committees

- Mubasharah Khalid Omer is a member of the organizing teams for the weekly PhD seminars at Inria Université Côte d'Azur and of the MOMI 2025 workshop.

10.1.2 Scientific events: selection

Chair of conference program committees

- Juliette Leblond was the chair of the scientific committee of the 11th conference on Inverse Problems, Control, and Shape Optimization (PICOF), Hammamet, Tunisia (October).

Member of the conference program committees

- Laurent Baratchart sat on the scientific committee of the Shanks conference, Nashville, Tennessee, USA (May).
- Sylvain Chevillard was a member of the program committee of the 32nd IEEE International Symposium on Computer Arithmetic (ARITH 2025) conference.

10.1.3 Journal

Member of the editorial boards

- Laurent Baratchart is a member of the editorial board of the journals *Computational Methods and Function Theory* (CMFT) and *Complex Analysis and Operator Theory* (CAOT).

10.1.4 Contributed talks

- Mubasharah Khalid Omer gave a talk at the 9th International Conference on Advanced COmputational Methods in ENgineering and Applied Mathematics (ACOMEN), Ghent, Belgium (September). She also presented a poster [21] at the 5th edition of Complex Days, Nice (February), see Section 6.1.

- Fatima Swaydan presented a poster [20] at the 11th conference on Inverse Problems, Control, and Shape Optimization, (PICOF), Hammamet, Tunisia (October).

10.1.5 Invited talks

- Laurent Baratchart was an invited speaker at the Shanks conference, Nashville, Tennessee, USA (May), and at Inverse Problems, Control, and Shape Optimization (PICOF), Hammamet, Tunisia (October). He was an invited speaker at the *Séminaire d'Analyse et Géométrie de l'Université de Provence*, Marseille Saint-Charles, October 20, 2025.

- Dmitry Ponomarev was an invited speaker at the Shanks conference, Nashville, Tennessee, USA (May).

10.2 Teaching - Supervision - Juries - Educational and pedagogical outreach

10.2.1 Teaching

- Sylvain Chevillard gives "Colles" (oral examination preparing undergraduate students for the competitive examination to enter French Engineering Schools) at Centre International de Valbonne (CIV) (2 hours per week).
  He contributed to the course *Environmental Issues* of Polytech Sophia Antipolis and addressed to all students of Polytech, whatever their pathway (level L3): the students had to attend several 1-hour conferences among a list of proposed conferences, and attend practical sessions (TP) where they would practically think about environmental questions (computation of their personal carbon footprint, introduction to the OpenLCA software to perform life-cycle analysis, participation to "Fresque du climat").
  He gave (6 times) a conference on Carbon Footprints and animated 6 hours of practical sessions with the OpenLCA software.

- Dmitry Ponomarev conducted tutorials ("travaux dirigés") for the course "Analyse 1", level L1, Université Côte d'Azur, January-April (38h).

- Fatima Swaydan was in charge of tutorials ("travaux dirigés") on advanced linear algebra, level L1, Université Côte d'Azur, September-December (32h).

10.2.2 Supervision

- PhD (not completed, see Section 6.2): Anass Yousfi, *Methods to estimate the net magnetic moment of rocks*, 2022-2025, advisors: Sylvain Chevillard, Juliette Leblond.

- PhD in progress: Mubasharah Khalid Omer, *Field preprocessing and treatment of complex samples in the paleo-magnetic context*, since October 2023, advisors: Juliette Leblond, Dmitry Ponomarev.

- PhD in progress: Fatima Swaydan, *Inverse magnetization problem in the paleomagnetic context*, since January 2025, advisors: Juliette Leblond, Dmitry Ponomarev.

- Internship: Axel Knecht, *Numerical methods for the problem of minimal norm equivalent source*, October 2025 to January 2026, advisors: Juliette Leblond, Dmitry Ponomarev.

- Internship: Rui Martins, October to December 2025, *Inverse problems in Magnetic Detection Electrical Impedance Tomography (MDEIT)*, advisors: Juliette Leblond, Dmitry Ponomarev.

10.2.3 Juries

- Juliette Leblond was a member of the examining committee for the defense of the PhD thesis of Anthony Gerber Roth, *On some geometric inverse problems*, Univ. Lorraine (IECL), Nancy (June).
  She was also a member of the “Comités de suivi individuels” (CSI) of the 1st year PhD for Laura Gee (Cronos team, ED STIC, July), David Tinoco (McTao team, ED SFA, October), Inria Université Côte d’Azur.

10.2.4 Educational and pedagogical outreach

- Sylvain Chevillard and Martine Olivi organized together with Luc Deneire, Sylvie Icart, Guillaume Urvoy-Keller (I3S) and Émilie Demoinet (Université Côte d’Azur), a one-day training program for PhD students (“Formation doctorale”) on the topic “Science, environment and society”.
- Juliette Leblond is teaching mathematics to teenagers (cycle 4) as volunteer at the “Collège Montessori Les Pouces Verts”, Mouans-Sartoux, since September (4h / week).

10.3 Popularization

- Juliette Leblond and Martine Olivi are members of Terra Numerica. In this capacity, they design workshops (geosciences, fractals, exponential growth, and eco-responsible digital technologies), and participate in several outreach events (science festivals, MathsC2+ training).

10.3.1 Productions (articles, videos, podcasts, serious games, ...)

- Martine Olivi was a member of the steering committee of the “Jeux à débattre : numérique et environnement”, the last debate game promoted by the association “L’arbre des connaissances”.

10.3.2 Participation in Live events

- Martine Olivi gave a lecture entitled “La croissance exponentielle a la côte” during the “Journées nationales de l’APMEP”, the association mathematics teachers in public schools.

10.3.3 Others science outreach relevant activities

- Martine Olivi, together with Aurélie Lagarrigue (Learning Lab), were the scientific facilitators of the citizen initiative with the municipality of Mouans-Sartoux (06).
  Such conventions are organized throughout France by the Alt IMPACT program, for the promotion of more responsible digital technologies. They are supervised by the CNRS, and bring together volunteer citizens, organizations, and scientific facilitators. The initial citizen initiative was launched with Grenoble Alpes Métropole at the end of 2024. In 2025, five new ones took place in French regions.

10.4 Community services

- Sylvain Chevillard is an elected member of the local board of the works council (AGOS) of Inria, with local treasurer duty. He benefits from an officially reduced working load of 30 hours per month for this purpose.
- Juliette Leblond is an elected member of the “Commission Administrative Paritaire (CAP)” and an associated member of the “Comité Égalité et Parité des chances” of Inria.

11 Scientific production

11.1 Major publications

[1] A. Arnold, S. Geevers, I. Perugia and D. Ponomarev. 'On the limiting amplitude principle for the wave equation with variable coefficients'. In: *Communications in Partial Differential Equations* 49.4 (26th Apr. 2024), pp. 333–380. DOI: 10.1080/03605302.2024.2341070. URL: https://hal.science/hal-03900877 (cit. on p. 13).

[2] L. Baratchart, A. Borichev and S. Chaabi. 'Pseudo-holomorphic functions at the critical exponent'. In: *Journal of the European Mathematical Society* 18.9 (2016), pp. 1919–1960. DOI: 10.4171/JEMS/634. URL: https://inria.hal.science/hal-00824224 (cit. on p. 8).

[3] L. Baratchart, S. Chevillard, A. Cooman, M. Olivi and F. Seyfert. 'Linearized Active Circuits: Transfer Functions and Stability'. In: *Mathematics in Engineering* 4.5 (12th Oct. 2021), pp. 1–18. DOI: 10.3934/mine.2022039. URL: https://hal.inria.fr/hal-01667606 (cit. on p. 21).

[4] L. Baratchart, S. Chevillard, D. P. Hardin, J. Leblond, E. A. Lima and J.-P. Marmorat. 'Magnetic moment estimation and bounded extremal problems'. In: *Inverse Problems and Imaging* 13.1 (Feb. 2019), p. 29. DOI: 10.3934/ipi.2019003. URL: https://hal.inria.fr/hal-01623991 (cit. on pp. 17, 27).

[5] L. Baratchart, S. Chevillard and T. Qian. 'Minimax principle and lower bounds in H^2-rational approximation'. In: *Journal of Approximation Theory* 206 (2015), pp. 17–47 (cit. on p. 13).

[6] L. Baratchart, S. Fueyo, G. Lebeau and J.-B. Pomet. 'Sufficient Stability Conditions for Time-varying Networks of Telegrapher's Equations or Difference Delay Equations'. In: *SIAM Journal on Mathematical Analysis* 53 (2021), pp. 1831–1856. DOI: 10.1137/19M1301795. URL: https://hal.inria.fr/hal-02385548 (cit. on p. 21).

[7] L. Baratchart, C. Villalobos Guillén, D. Hardin, M. Northington and E. Saff. 'Inverse Potential Problems for Divergence of Measures with Total Variation Regularization'. In: *Foundations of Computational Mathematics* 20 (2020), pp. 1273–1307. DOI: 10.1007/s10208-019-09443-x. URL: https://hal.inria.fr/hal-02424672 (cit. on pp. 15, 17).

[8] M. Darbas, J. Leblond, J.-P. Marmorat and P.-H. Tournier. 'Numerical resolution of the inverse source problem for EEG using the quasi-reversibility method'. In: *Inverse Problems* 39.11 (2023), p. 115003. DOI: 10.1088/1361-6420/acf9c6. URL: https://inria.hal.science/hal-03880526 (cit. on p. 15).

[9] M. Olivi, F. Seyfert and J.-P. Marmorat. 'Identification of microwave filters by analytic and rational H2 approximation'. In: *Automatica* 49.2 (15th Jan. 2013), pp. 317–325. DOI: 10.1016/j.automatica.2012.10.005. URL: https://hal.inria.fr/hal-00753824 (cit. on p. 12).

11.2 Publications of the year

International journals

[10] L. Baratchart, S. Fueyo and J.-B. Pomet. 'Exponential stability of linear periodic difference-delay equations'. In: *SIAM Journal on Mathematical Analysis* 57 (2025), pp. 3110–3145. DOI: 10.1137/23M160133X. URL: https://inria.hal.science/hal-03500720 (cit. on p. 21).

[11] L. Baratchart, H. Haddar and C. Villalobos Guillén. 'Silent sources on a surface for the Helmholtz equation and decomposition of L^2 vector fields'. In: *SIAM Journal on Mathematical Analysis* 57.1 (2025), pp. 682–713. DOI: 10.1137/23M1626578. URL: https://hal.science/hal-04367726 (cit. on p. 11).

[12] L. Baratchart, J. Leblond and M. Nemaire. 'Silent sources in L^p and Helmholtz-type decompositions'. In: *SIAM Journal on Mathematical Analysis* 57.3 (2025), pp. 2715–2745. DOI: 10.1137/24M163033. URL: https://inria.hal.science/hal-03915548 (cit. on p. 10).

[13] D. Ponomarev. 'Magnetisation Moment of a Bounded 3D Sample: Asymptotic Recovery from Planar Measurements on a Large Disk'. In: *Journal of Computational and Applied Mathematics* (2025). DOI: 10.1016/j.cam.2025.117085. URL: https://hal.science/hal-03813559. In press (cit. on p. 26).

Scientific book chapters

[14] D. Ponomarev. 'A method to extrapolate the data for the inverse magnetisation problem with a planar sample'. In: *Inverse Problems: Modeling and Simulation - Extended Abstracts of the IPMS Conference 2024*. 18th July 2025. DOI: 10.1007/978-3-031-87213-6. URL: https://hal.science/hal-05393788.

Reports & preprints

[15] L. Baratchart, D. P. Hardin and C. Villalobos Guillén. *Notes on the discretization of TV-NORM regularized inverse potential problems*. 27th Mar. 2025. URL: https://hal.science/hal-05008293.

[16] L. Baratchart and S. Chevillard. *Lower bounds in H^2-rational approximation to z^N*. Jan. 2025. URL: https://inria.hal.science/hal-05385529.

[17] S. Chevillard and J. Leblond. *Some Explicit Integrals Related to the Remanent Magnetization Inverse Problem*. RT-0525. Inria, Nov. 2025, p. 23. URL: https://inria.hal.science/hal-05343567 (cit. on p. 27).

[18] S. Chevillard, J. Leblond and A. Yousfi. *Asymptotic integrals, on the disk, of the magnetic field generated by a remanent magnetization*. RR-9601. Inria, Nov. 2025. URL: https://hal.science/hal-05361820 (cit. on p. 26).

[19] S. Fueyo, L. Baratchart and J.-B. Pomet. *Monodromy Structure and Harmonic Transfer Function in Lossless Transmission Line Circuits*. 2025. URL: https://hal.science/hal-05042937.

Other scientific publications

[20] J. Leblond, D. Ponomarev and F. Swaydan. 'Total magnetisation reconstruction: Impact of the measurement direction of the magnetic field on the quality of the reconstruction'. In: PICOF 2025 - 11th International Conference on Inverse Problems, Control and Shape Optimization. Hammamet, Tunisia, 28th Oct. 2025. URL: https://hal.science/hal-05363534 (cit. on pp. 27, 31).

[21] M. K. Omer, J. Leblond and D. Ponomarev. 'Field Extrapolation in Inverse Magnetization Problems'. In: Complex Days, 2025. Nice (06000), France, 6th Feb. 2025. URL: https://hal.science/hal-05368204 (cit. on pp. 22, 31).

11.3 Cited publications

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[23] A. Arnold, S. Geevers, I. Perugia and D. Ponomarev. 'An adaptive finite element method for high-frequency scattering problems with smoothly varying coefficients'. In: *Computers & Mathematics with Applications* 109 (2022), pp. 1–14 (cit. on p. 13).

[24] A. Arnold, S. Geevers, I. Perugia and D. Ponomarev. 'On the exponential time-decay for the one-dimensional wave equation with variable coefficients'. In: *Communications on Pure and Applied Analysis* 21.10 (2022), p. 3389 (cit. on p. 14).

[25] P. Asensio. 'Inverse problems of source localization with applications to EEG and MEG'. Université Côte d'Azur, 21st Sept. 2023. URL: https://hal.science/tel-04366056 (cit. on pp. 13, 15, 20, 28).

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[29] L. Baratchart, M. Cardelli and M. Olivi. 'Identification and rational L^2 approximation: a gradient algorithm'. In: *Automatica* 27 (1991), pp. 413–418 (cit. on p. 12).

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[32] L. Baratchart, Y. Fischer and J. Leblond. 'Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation'. In: *Complex Variables and Elliptic Equations* (2014), p. 41. DOI: 10.1080/17476933.2012.755755. URL: https://hal.archives-ouvertes.fr/hal-00909577 (cit. on p. 8).

[33] L. Baratchart, S. Fueyo and J.-B. Pomet. 'Integral representation formula for linear non-autonomous difference-delay equations'. In: *Journal of Integral Equations and Applications* 36.4 (2024). DOI: 10.1216/jie.2024.36.407. URL: https://hal.science/hal-04204500 (cit. on p. 22).

[34] L. Baratchart, C. Gerhards and A. Kegeles. 'Decomposition of L2-vector fields on Lipschitz surfaces: characterization via null-spaces of the scalar potential'. In: *SIAM Journal on Mathematical Analysis* 53.4 (2021), pp. 4096–4117. DOI: 10.1137/20M1387754. URL: https://inria.hal.science/hal-03086446 (cit. on p. 10).

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