2025 *Activity Report*

RESEARCH CENTRE: **Inria Saclay Centre**

Project-Team

MATHEXP

Computer algebra, experimental mathematics, and interactions

Project-Team MATHEXP

Creation of the Project-Team: 2022 April 01

Each year, Inria research teams publish an Activity Report presenting their work and results over the reporting period. These reports follow a common structure, with some optional sections depending on the specific team. They typically begin by outlining the overall objectives and research programme, including the main research themes, goals, and methodological approaches. They also describe the application domains targeted by the team, highlighting the scientific or societal contexts in which their work is situated. The reports then present the highlights of the year, covering major scientific achievements, software developments, or teaching contributions. When relevant, they include sections on software, platforms, and open data, detailing the tools developed and how they are shared. A substantial part is dedicated to new results, where scientific contributions are described in detail, often with subsections specifying participants and associated keywords. Finally, the Activity Report addresses funding, contracts, partnerships, and collaborations at various levels, from industrial agreements to international cooperations. It also covers dissemination and teaching activities, such as participation in scientific events, outreach, and supervision. The document concludes with a presentation of scientific production, including major publications and those produced during the year.

Keywords

Computer sciences and digital sciences

A8.1. – Discrete mathematics, combinatorics

A8.3. – Geometry, Topology

A8.4. – Computer Algebra

A8.5. – Number theory

Other research topics and application domains

B9.5.2. – Mathematics

B9.5.3. – Physics

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1 Team members, visitors, external collaborators

Research Scientists

- Frédéric Chyzak [Team leader, INRIA, Senior Researcher, HDR]
- Philippe Dumas [Éducation Nationale, retired]
- Guy Fayolle [INRIA, Emeritus]
- Pierre Lairez [INRIA, Researcher]

Post-Doctoral Fellows

- Ricardo Thomas Buring [INRIA, Post-Doctoral Fellow, until Aug 2025]
- Claudia Fevola [INRIA, Post-Doctoral Fellow]
- Rafael Mohr [INRIA, Post-Doctoral Fellow, until Sep 2025]

PhD Students

- Hadrien Brochet [INRIA]
- Alexandre Goyer [Éducation Nationale, until Oct 2025]
- Alexandre Guillemot [INRIA]
- Théo Ternier [INRIA, from Sep 2025]

Interns and Apprentices

- Jaali Mazzaggio [INRIA, Intern, from Mar 2025 until Aug 2025]
- Théo Ternier [INRIA, Intern, from Feb 2025 until Aug 2025]

Administrative Assistants

- Bahar Carabetta [INRIA, until Nov 2025]
- Ekaterina George [INRIA, from Nov 2025]

2 Overall objectives

“Experimental mathematics” is the study of mathematical phenomena by computational means. “Computer algebra” is the art of doing effective and efficient exact mathematics on a computer. The MATHEXP team develops both themes in parallel, in order to discover and prove new mathematical results, often out of reach for classical human means. It is our strong belief that modern mathematics will benefit more and more from computer tools. We ambition to provide mathematical users with appropriate algorithmic theories and implementations.

Besides the classification by mathematical and methodological axes to be presented in §3, MATHEXP’s research falls into four interconnected categories, corresponding to four different ways to produce science. The raison d’être of the team is solving core questions that arise in the practice of experimental mathematics. Through the experimental mathematics approach, we aim at applications in diverse areas of mathematics and physics. All rests on computer algebra, in its symbolic and seminumerical aspects. Lastly, software development is a significant part of our activities, with the aim of enabling cutting-edge applications and disseminating our tools. Each of these four levels is reflected in the thematic axes of the research program.

2.1 Experimental mathematics

In science, observation and experiment play an important role in formulating hypotheses. In mathematics, this role is shadowed by the primacy of deductive proofs, which turn hypotheses into theorems, but it is no less important. The art of looking for patterns, of gathering computational evidence in support of mathematical assertions, lies at the heart of experimental mathematics, promoted by Euler, Gauss and Ramanujan. These prominent mathematicians spent much of their time doing computations in order to refine their intuitions and to explore new territories before inventing new theories. Computations led them to plausible conjectures, by an approach similar to those used in natural sciences. Nowadays, experimental mathematics has become a full-fledged field, with prominent promoters like Bailey and Borwein. In their words [39], experimental mathematics is "the methodology of doing mathematics that includes the use of computation for

- gaining insight and intuition,
- discovering new patterns and relationships,
- using graphical displays to suggest underlying mathematical principles,
- testing and especially falsifying conjectures,
- exploring a possible result to see if it is worth formal proof,
- suggesting approaches for formal proof,
- replacing lengthy hand derivations with computer-based derivations,
- confirming analytically derived results."

2.2 Foundations of computer algebra

At a fundamental level, we manipulate several kinds of algebraic objects that are characteristic of computer algebra: arbitrary-precision numbers (big integers and big floating-point numbers, typically with dozens of thousands of digits), polynomials, matrices, differential and recurrence operators. The first three items form the common ground of computer algebra [79]. We benefit from years of research on them and from broadly used efficient software: general-purpose computer-algebra systems like Maple, Magma, Mathematica, Sage, Singular; and also special-purpose libraries like Arb, Fgb, Flint, Msolve, NTL. Current developments, whether software implementation, algorithm design or new complexity analyses, directly impact us. The fourth kind of algebraic objects, differential and recurrence operators, is more specific to our research and we concentrate our efforts on it. There, we try to understand the basic operations in terms of computational complexity. Complexity is also our guide when we recombine basic operations into elaborate algorithms. In the end, we want fast implementations of efficient algorithms.

Here are some of the typical questions we are interested in:

- Do some of the solutions of a linear ordinary differential equation (ODE) satisfy a simpler ODE? This relates to the problem of factoring differential operators.
- Is a given linear partial differential equation (PDE) a consequence of a set of other PDEs? This relates to the problem of computing Gröbner bases in a differential setting.
- Given a solution $f(x,y)$ of a system of linear PDEs, how to compute differential equations for $f(x,0)$ or $\int_0^1 f(x,y)\mathrm{d}y$? This falls into the realm of symbolic integration questions.
- Given a linear ODE with initial condition at 0, how to evaluate numerically the unique solution at 1 with thousands of digits of precision? This is the gist of our seminumerical methods.

2.3 Applications

Getting involved in applications is both an objective and a methodology. The applications shape the tools that we design and foster their dissemination.

Combinatorics is a longstanding application of computer algebra, and conversely, computer algebra has a deep impact on the field. The study of random walks in lattices, first motivated by statistical physics and queueing theory, features prominent examples of experimental mathematics and computer-assisted proofs. Our main collaborators in combinatorics are Mireille Bousquet-Mélou (Université de Bordeaux), Stephen Melczer (University of Waterloo) and Kilian Raschel (Université d'Angers).

Probability theory. Apart from the already mentioned interest in random walks, which is a classical topic in probability theory, and on which we have an expert, Guy Fayolle, in our group, the main applications we have in mind are to integrals arising from: 2D fluctuation theory (generalizing arc-sine laws in 1D); moments of the quadrant occupation time for the planar Brownian motion; persistence probability theory (survival functions of first passage time for real stochastic processes); volumes of structured families of polytopes also arising in polyhedral geometry and combinatorics. Our main interactions on these topics are with Gerold Alsmeyer (U. Münster), Dan Betea (KU Leuven), and Thomas Simon (U. Lille).

Number theory, and especially *diophantine approximation*, are also fields with longstanding users of computer algebra tools. For example, the recently discovered sequence of integrals

$$\int_{4-2i}^{4+2i} \frac{(x-4+2i)^{4n}(x-4-2i)^{4n}(x-5)^{4n}(x-6+2i)^{4n}(x-6-2i)^{4n}}{x^{6n+1}(x-10)^{6n+1}}\,\mathrm{d}x, \quad n \geq 0,$$

whose analysis leads to the best known measure of irrationality of π, can hardly be found by hand [129]. Yet, the discovery and the proof of such a result requires sophisticated tools from experimental mathematics. Our main collaborators in number theory are Boris Adamczewski (Université Lyon 1), Xavier Caruso (Université de Bordeaux), Stéphane Fischler (Université Paris Saclay), Tanguy Rivoal (Université Grenoble Alpes), Wadim Zudilin (University Nijmegen). Mahler equations are other aspects of number theory, in relation to *automata theory*, and appear in several of our research axes. Philippe Dumas, in our group, and Boris Adamczewski, already mentioned, have long been experts in this topic.

In *algebraic geometry*, in spite of tremendous theoretical achievements, it is a challenge to apply general theories to specific examples. We focus on putting into practice transcendental methods through symbolic integration and seminumerical methods. Our main collaborators are Emre Sertöz (Max Planck Institute for Mathematics) and Duco van Straten (Gutenberg University).

In *statistical physics*, the *Ising model*, and its generalization, the *Potts model*, are classical in the study of phase transitions. Although the Ising model with no magnetic field is one of the most important exactly solved models in statistical mechanics (Onsager won the Nobel prize 1968 for this), its *magnetic susceptibility* continues to be an unsolved aspect of the model. In absence of an exact closed form, the susceptibility is approached analytically, via the singularities of some multiple integrals with parameters. Experimental mathematics is a key tool in their study. Our main collaborators are Jean-Marie Maillard (SU, LPTMC) and Tony Guttmann (U. Melbourne).

In *quantum mechanics*, turning theories into predictions requires the computation of Feynman integrals. For example, the reference values of experiments carried out in particle accelerators are obtained in this way. The analysis of the structure of Feynman integrals benefits from tools in experimental mathematics. Our main collaborator in this field is Pierre Vanhove (CEA, IPhT).

2.4 Software

We ambition to provide efficient software libraries that perform the core tasks that we need in experimental mathematics. We target especially four tasks of general interest: algebraic algorithms for manipulating systems of linear PDEs, univariate and multivariate guessing, symbolic integration, and seminumerical integration.

For several reasons, we want to stay away from a development model that is too tied to commercial computer algebra systems. Firstly, they restrict dissemination and interoperability. Secondly, they do not offer the level of control that we need to implement these foundations efficiently. Concretely, we will develop open-source libraries in C++ for the most fundamental tasks in our research area. Computer algebra systems, like Sagemath or Maple, are good at coordinating primitive algorithms, but too high-level to implement them efficiently. We seek solid software foundations that provide the primitive algorithms that we need. This is necessary to implement the new higher-level algorithms that we design, but also to reach a performance level that enables new applications. Still, we will strive to expose our libraries to the prominent computer-algebra systems, especially Maple and Sagemath, used by many colleagues.

Besides, there is a growing interest in the programming language Julia for computer algebra, as shown by the Oscar project. We already internally use Julia and occasionally some of the libraries Oscar is build upon, and we want to promote this young ecosystem. It is very attractive to contribute to it, but on the flip side of the coin, it is too young to offer the same usability as Maple, or even Sagemath. So there is an assumed element of risk taking in our intent to also make our libraries available to Julia.

3 Research program

3.1 Algebraic algorithms for multivariate systems of equations

At large, MATHEXP deals with algebraic and seminumerical methods. This part goes through the fundamental aspects of the algebraic side. As opposed to numerical analysis where numerical evaluations underlie the basic algorithms, algebraic methods manipulate functions through functional equations. Depending on the context, different kinds of functional equations are appropriate. Algebraic functions are handled through polynomial equations and the classical theory of polynomial systems. To deal with integrals, systems of linear partial differential equations (PDEs) are appropriate. In combinatorics and number theory appears the need for non-linear ordinary differential equations (ODEs). We also consider other kinds of functional equations more related to discrete structures, namely linear recurrence relations, q-analogues and Mahler equations.

The various types of functional equations raise similar questions: is a given equation consequence of a set of other equations? What are the solutions of a certain type (polynomial, rational, power series, etc.)? What is the local behavior of the solutions? Algorithms to solve these problems support an important part of our research activity.

3.1.1 Holonomic systems of linear PDEs

One of the major data structure that we consider are systems of linear PDEs with polynomial coefficients. A system that has a finite dimensional solution space is called *holonomic* and a function that is solution of a holonomic system is called holonomic too. The theory of holonomy is important because it allows for an algebraic theory of analysis and integration (on this aspect see also §3.2). The basic objects of holonomy theory are linear differential operators, that are some sort of quasicommutative polynomials, and ideals in rings of linear differential operators, called *Weyl algebras*. In this aspect, holonomy theory is analogue to the theory of polynomial systems, where the basic objects are commutative polynomials and ideals in polynomial rings. Some of the important concepts, for example the concept of Gröbner basis, are also similar. Gröbner bases are a way to describe all the consequences of a set of equations.

As much as Gröbner bases in polynomial rings are the backbone of effective commutative algebra, Gröbner bases in Weyl algebras of differential operators are the backbone of effective holonomy theory, which includes integration. In a commutative setting, there has been a long way from the early work of Buchberger to today's state-of-the-art polynomial system solving libraries [31]. We will develop a similar enterprise in the noncommutative setting of Weyl algebras. It will unlock a lot of applications of holonomy theory.

Following the commutative case, progress in a differential context will come from an appropriate theory and efficient data structures. We will first develop a matrix approach to handle simultaneous reduction of differential operators as the F4 algorithm for the polynomial case [74]. The real challenge here is more practical than theoretical. It is not difficult to come with *some* F4 algorithm in the differential case. But will it be efficient? From the experience of modern Gröbner engines in the commutative case, we know that efficient implementation of simultaneous reduction requires a significant amount of low-level programming to deal with sparse matrices with a special structure. We also know that many choices, irrelevant to the mathematical theory, strongly influence the running times. The noncommutativity of differential operators adds extra complications, whose consequences are still to be understood at this level. We want to reuse, as much as possible, the specialized linear algebra libraries that have been developed in the polynomial context [50, 31], but we may have to elude the densification of products induced by noncommutativity.

On a more theoretical aspect, one step further in the analysis is that the possible analogues of the F5 algorithm [75] are not fully explored in a differential setting. We may expect not only faster algorithms, but also new algorithms for operating on holonomic functions (Weyl closure for example, see §3.1.2). Rafael Mohr started a PhD thesis in the team on using F5 for computing equidimensional decompositions in the commutative case.

3.1.2 Desingularization of PDEs

Among the structural properties of systems of linear differential or difference equations with polynomial coefficients, the question of understanding and simplifying their singularity structure pops up regularly. Indeed, an equation or a system of equations may exhibit singularities that no solution have, which are then called apparent singularities. Desingularization is a process of simplifying a ∂-finite system by getting rid of its apparent singularities. This is done at the cost of increasing the order of equations, thus, the dimension of their solution space. The univariate setting has been well studied over time, including in computer algebra for its computational aspects [19, 18]. This led to the notion of order-degree curve [56, 57, 55]: a given function can cancel an ODE or ORE (ordinary recurrence equation) of small order with a certain coefficient degree, and also other ODEs or OREs of higher orders, possibly with smaller coefficient degrees. In certain applications, the ODE or ORE of minimal order may be too large to be obtained by direct calculations. It appears that the total size of the equations, that is, the product of order by degree, can be more relevant to optimize the speed of algorithms. This is a phenomenon that we observed first in relation to algebraic series [44], and we want to promote further this idea of trading minimality of order for minimality of total size, with the goal of improved speed. On the other hand, apparent singularities have been defined only recently in the multivariate holonomic case [58].

Our project includes developing good notions and fast heuristic methods for the desingularization of a ∂-finite system, first in the differential case, where it is expected to be easier, then in the case of recurrence operators.

Moreover, fast algorithms will be obtained for testing the separability of special functions: in a nutshell, this problem is to decide whether the solutions to a given system also satisfy linear differential or difference equations in a single variable, and algorithmically this corresponds to obtaining structured multiples of operators with a structure similar to that for desingularization.

In the multivariate case, the operation of saturating an ideal in the Weyl algebra by factoring out (and removing) all polynomial factors on the left is known under the name of Weyl closure. This relates to desingularization as the Weyl closure of an ideal contains all desingularized operators. Weyl closure also is a relative of the radical of an ideal in commutative algebra: given an ideal of linear differential operators, its Weyl closure is the (larger) ideal of all operators that annihilate any function solution to the initial ideal. Computing Weyl closure applies to symbolic integration, and algorithms exist to compute it [126, 125], although they are slow in practice. Weyl closure also plays an important role in applications to the theory of special functions, e.g., in the study of GKZ-systems (a.k.a. A-hypergeometric systems) [104], and in relation to Fischer distribution and maximum likelihood estimation in statistics [22, 80]. Algorithms for Weyl closure should then be obtained, by basing on desingularization as a subtask.

3.1.3 Well-foundedness of divide-and-conquer recurrence systems

Converting a linear Mahler equation with polynomial coefficients (see §3.3.3) into a constraint on the coefficient sequence of its series solutions results in a recurrence between coefficients indexed with rational numbers, which must be interpreted to be zero at noninteger indices. The recurrence can be replaced with a system of recurrences by cases depending on residues modulo some power of the base b. The literature also alternatively introduces recurrences with indices expressed with floor/ceiling functions, typically so for fine complexity analysis of divide-and-conquer algorithms. For sequences that can be recognized by automata ("automatic sequences") and their generalizations ("b-regular sequences"), it is natural to consider a system of recurrences on several sequences, with a property of closure under certain operations of taking subsequences: restricting to even indices, or odd indices, or more generally indices with a given residue modulo the base b. This variety of representations calls for algorithms to be able to convert from one another, to check the consistency of a given system of recurrences, and to identify those terms of the sequence that determine all others (which are typically not just a few first terms). In the continuation of [62] that developed a Gröbner-bases theory as a pre-requisite for this goal, we will address those problems of conversion and well-foundedness.

3.1.4 Software

Software development is a real challenge, regarding the symbolic manipulation of linear PDEs. While symbolic integration has gained more and more recognition, its execution is still reserved to experts. Providing a highly efficient software library with functionalities that come as close as possible to the actual integrals, rather than some idealized form, will foster adoption and applications. In the past, the lack of solid software foundations has been an obstacle in implementing newly developed algorithms and in disseminating our work. It was the case, for example, for our work on binomial sums [46], or the computation of volumes [98], where having to use an integration algorithm implemented in Magma has been a major obstacle.

What is lacking is a complete tool chain integrating the following three layers:

1. the computation of Gröbner bases of holononomic systems, as discussed in §3.1.1;
2. the basic algorithms for manipulating holonomic systems, such as the desingularization discussed in §3.1.2 but also the classical aspects of symbolic integration;
3. the algorithms relevant for applications, including all the aspects covered in §3.2.

The first layer of the toolchain will be developed in C++ for performance but also to open the way to an integration in free computer algebra systems, like Sagemath or Macaulay2. We will benefit from years of experience of the community and close colleagues in implementing Gröbner basis algorithms in the commutative case. The third layer of the toolchain should be easily accessible for the users, so at least available in Sagemath. Some of our current software development, related to the second layer, already happens in Julia (as part of R. Mohr's PhD work).

3.2 Symbolic integration with parameters

Among common operations on functions, integration is the most delicate. For example, differentiation transforms functions of a certain kind into functions of the same kind; integration does not. For this reason, integration is also *expressive*: it is an essential tool for defining new functions or solving equations, not to mention the ubiquitous Fourier transform and its cousins. Integration is the fundamental reason why holonomic functions are so important: integrals of holonomic functions are holonomic. Algorithms to perform this operation enable many applications, including: various kinds of coefficient extractions in combinatorics, families of parametrized integrals in mathematical physics, proofs of irrationality in number theory, and computations of moments in optimization.

Given a function $F(\mathbf{t}, \mathbf{x})$ of two blocks of variables $\mathbf{t} = t_1, \dots, t_s$ and $\mathbf{x} = x_1, \dots, x_n$, and an integration domain $\Omega(\mathbf{t}) \subseteq \mathbb{R}^n$, how to compute the function

$$G(\mathbf{t}) = \int_{\Omega(\mathbf{t})} F(\mathbf{t}, \mathbf{x}) \mathrm{d}\mathbf{x}?$$

Concretely, $F(\mathbf{t}, \mathbf{x})$ is described by a system of linear PDEs with polynomial coefficients, $\Omega(\mathbf{t})$ is given by polynomial inequalities, and we want a system of PDEs describing $G(\mathbf{t})$. Note here the presence of parameters which makes it possible to describe the result of integration with PDEs. When there are no parameters, the result is a numerical constant. Even though we deal with them in an entirely different way (see §3.5), we still mostly rely on symbolic integration with parameters.

From the algebraic and computational point of view, integration has several analogues. Discrete sums are the prominent example, but there are also q-analogues, Mahlerian functions, and some others. At large, algorithms for symbolic integration, or its analogues, perform a sort of elimination in a ring of differential operators. There are some links with elimination theory and related algorithms as developed for the study of polynomial systems of equations.

Symbolic integration is an historical focus of MATHEXP's founding members with many significant contributions. Compared to our previous activities, we want to put more emphasis on software development. We are at a point where the theory is well understood but the lack of efficient implementations hinders many applications. Naturally, this effort will rest on the results obtained in §3.1.

3.2.1 Integrals with boundaries

The algebraic aspects of symbolic integration are best understood when the integration domain has no boundary: typically $\mathbb{R}^n$ or a topological cycle in $\mathbb{C}^n$. Indeed, in this context we have the so-called *telescopic relation* which states that the integral of a derivative vanishes: for example, if $H(\mathbf{t}, \mathbf{x})$ is rapidly decreasing, then

$$\int_{\mathbb{R}^n} \frac{\partial H}{\partial x_i} \mathrm{d}\mathbf{x} = 0.$$

It gives a nice algebraic flavor to the problem of symbolic integration and reduces it to the study of the quotient space $\mathcal{F} / \big(\frac{\partial}{\partial x_1} \mathcal{F} + \cdots + \frac{\partial}{\partial x_n} \mathcal{F} \big)$, where $\mathcal{F}$ is a suitable function space containing the integrand. A large part of the algorithms developed so far focuses on this case. Yet, many applications do not fit in this idealized setting. For example, Beukers' proof of the irrationality of $\zeta(3)$ [34] uses the two integrals

$$\oiiint_{\gamma} R \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \text{ and } \iiint_{[0,1]^3} R \,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \text{ where } R(t, x, y, z) = \frac{1}{1-(1-xy)z-txyz(1-x)(1-y)(1-z)}.$$

The first one, where the integration domain is some complex cycle γ, is well handled by current algorithms. The second is not, and this is unsatisfactory for further applications of symbolic integration in number theory. In this particular case, we may think of an algorithm that would reduce the integration on the cube to an integration without boundary and an integration on the boundary of the cube. This boundary just consists of 6 squares, which calls for a recursive procedure. Unfortunately, the integration domain touches the poles of the integrand, so operations like integrating only part of a function or integration by parts or differentiation under the integral sign may not be meaningful by lack of integrability. It is not known how to deal with this issue automatically. For more general domains of integration, it is not even clear what kind of recursive procedure can be applied.

The next generation of symbolic integration algorithms must deal with integrals defined on domains with boundaries. The framework of algebraic D-modules seems to be very appropriate and already features some algorithms. But this is not the end of the story, as this line of research has not led yet to efficient implementations. We identified two ways of action to reach this goal. Firstly, existing algorithms [110, 111] put too much emphasis on computing a minimal-order equation for the integral. While this is an interesting property, other kinds of integration algorithms have successfully relaxed this condition. For example, for integrating rational functions, the state-of-the-art algorithm [97] depends on a parameter $r > 0$. The computed equation is minimal only for r large enough, which corresponds to the degeneration rank of some spectral sequence [68]. In practice, this has never been an obstacle: most of the time we obtain a minimal equation with a small value of r. For the few remaining cases, we will soon propose a generalized procedure to minimize the equation *a posteriori*; this will be a consequence of a work on univariate guessing (see §3.4.1) that bases and expands on [47]. The algorithm by small values of r applicable in most cases already outperforms previous ones in terms of computational complexity [45] and practical performance, being able to compute integrals that were previously out of reach. We consider it to be a special case of the general algorithm that we want to develop, and a proof of feasibility. However, the effort will be vain without significant progress on the computation of Gröbner bases in Weyl algebras. Fortunately, and this is the second way of action, we think that the framework of algebraic D-modules enables efficient data structures modeled on recent progress in the context of polynomial systems. Progress in this direction (as explained in §3.1.1) will immediately lead to significant improvement for symbolic integration.

3.2.2 Reduction-based creative telescoping

The approach to symbolic integration based on *creative telescoping* is a definite expertise of the team. Although the approach is difficult to use for integrals with boundaries, it still has many appeals. In particular, it generalizes well to discrete analogues. Recently, the team has initiated the development of a new line of algorithms, called *reduction-based*. After continuing work, this line has not yet been extended to full generality [43, 89]. These recent theoretical developments are not yet reflected in current software packages (only prototype implementations exist) and therefore their practical applicability, and how the algorithms compare, is not yet fully understood. Filling these gaps will be a good starting point for us, but the ultimate goal will be to formulate analogue algorithms for the difference case (summation of holonomic sequences), for the q-case, and for general mixed cases. We expect that these advances in the theory will have a great impact on various applications.

3.2.3 Holonomic moment methods

In applied mathematics, the method of moments provides a computational approach to several important problems involving polynomial functions and polynomial constraints: polynomial optimization, volume estimation, computation of Nash equilibria, ... [101]. This method considers infinite-dimensional linear optimization problems over the space of Borel measures on some space $\mathbb{R}^n$. They admit finite-dimensional relaxations in terms of linear matrix inequalities where a measure μ is represented approximately by a finite number of moments $\int \mathbf{x}^\alpha d\mu$.

From the holonomic point of view, the generating function of the moments of μ — or, equivalently, the characteristic function $\phi(\mathbf{u}) = \int \exp(i\mathbf{u} \cdot \mathbf{x}) d\mu$ — is holonomic for a large class of measures μ (which includes all measures that appear in current applications of the method of moments). This remark already unlocks some applications where the current bottleneck is the computation of many moments: differential equations on $\phi(\mathbf{u})$ reflect recurrence relations on the moments, and computing the former with symbolic integration will lead to efficient algorithms for computing the moments.

A line of research developed recently [92, 102, 100, 121, 103] focuses on reducing the size of the matrices in the linear matrix inequalities (LMI) involved in the relations by using *pushforward measures*. For example, let us consider a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ and the problem of computing the volume of $\{p \in [0,1]^n \mid f(p) \geq 0\}$. The article [86] solves this problem with a linear program over Borel measures on $[0,1]^n$. Using the pushforward measure, the work [92] reduces to a linear program over measures on $\mathbb{R}$, supposedly much easier to solve. However, this comes at the cost of computing the moments $\mu_k = \int_{[0,1]^n} f^k dx_1 \cdots dx_n$ for increasingly large values of k. While this is an elementary task (it is enough to expand f^k), the number of monomials to compute is $\sim \frac{1}{n!}(k \deg(f))^n$, for large k, and this becomes the bottleneck of the method. The computation of the generating series $\sum_{k \geq 0} \mu_k t^k$ using symbolic integration enables the computation of a linear recurrence relation, of size $\deg(f)^n$ at most, for the moments μ_k, and we can compute the $\mu_0, \dots, \mu_k$ in $O(k \deg(f)^n)$ arithmetic operations only, or $\tilde{O}(k^2 \deg(f)^n)$ bit operations. This should be a low-hanging fruit as soon as we have reasonable implementations of symbolic integration on domains with boundaries (see §3.2.1). Naturally, the constant in the big O hides the size of the ODE of which the generating function is solution, and it may be exponential in n. But this is only a worst-case bound, and any nongeneric geometric property will tend to make this ODE smaller.

One step further, we will try to interpret the whole moment method in the holonomic setting. The differential equation for $\sum_{k \geq 0} \mu_k t^k$ not only enables the computation of the moments μ_k, it somehow encodes *all* the μ_k. Recovering numerical values, such as the volume, from this differential equation is akin to the seminumerical algorihtms we know (see §3.5.1). As a next step, we will study how some optimization problems treated by the method of moments behave in this holonomic setting. We think especially of the problems of chance optimization and chance constrained optimization [91]: in the former, one maximizes the probability of success over the design parameters; in the latter one optimizes a goal while ensuring that some probability remains low.

3.2.4 New aspects of symbolic integration

Mahlerian telescopers. Here the aim is to determine the relations satisfied by a solution of a Mahler equation (see §3.3.3). A natural generalization is to search for relations among solutions of different Mahler equations. Our objective is to provide an algorithmic answer to this generalization, for (Laurent) power series y_i solutions of inhomogeneous first-order equations of the form $y_i(z^p) + a_i(z)y_i(z) = b_i(z)$, with coefficients in $\bar{\mathbb{Q}}(z)$. We will start with the easy case where all a_i's are equal to 1. Under this assumption, a theorem of Hardouin and Singer guarantees that there exists an algebraic relation with coefficients in $\bar{\mathbb{Q}}(z)$ between $y_1, \dots, y_n$ if and only if there exists a Mahlerian telescoper between the $b_i(z)$. (This originates in Hardouin's PhD thesis and was generalized in [84].) We will work on making algorithmic such an existence test, and if possible the calculation of such telescopers. For this, we will be inspired by existing algorithms for calculating telescopers for other types of functional operators.

D-algebraicity and elliptic telescoping. Random walks confined to the quarter plane is a well studied topic, as testified by the book [76]. A new algebraic approach, relying on the Galois theory of difference equations, has been introduced in [70] to determine the nature of the generating series of such walks. This approach gives access to the D-algebraicity of the generating functions, that is, to the knowledge of whether they satisfy some differential equations (linear or non-linear). More precisely, D-algebraicity is shown to be equivalent to the fact that a certain telescopic equation, similar to the one appearing in the classical context of creative telescoping, but defined on an elliptic curve attached to the walk model, admits solutions in the function field of that curve. For the moment, the corresponding telescoping equations are solved by hand, in a quite ad-hoc fashion, using case-by-case treatment. We aim at developing a systematic and automatized approach for solving this kind of *elliptic creative telescoping* problems. To this end, we will import and adapt our algorithmic methods from the classical case to the elliptic framework.

3.2.5 Software

Because of the dependency of the software pertaining to symbolic integration on developments on multivariate systems, our goals related to software on symbolic integration have been described in §3.1.4.

3.3 Computerized classification of functions and numbers

Classifying objects, determining their nature, is often the culmination of a mature theory. But even the best established theories can be impracticable on a concrete instance, either by a lack of effectiveness or by a computational barrier. In both cases, an *algorithm* is missing: we have to systematize, but also *effectivize and automate efficiently*. This is what we propose to do, in order to solve classification problems relating to numbers, analytical functions, and combinatorial generating series.

3.3.1 Practical tests of algebraicity and transcendence for holonomic functions

It is an old question addressed by Fuchs in the 1870s of whether one can decide if *all* solutions of a given linear differential equation are algebraic. Singer showed in [120] that there exists an algorithm which takes as input a linear differential equation with coefficients in $\mathbb{Q}[x]$, and decides in a finite number of steps whether or not it has a full basis of algebraic solutions. If the answer is negative, this does not automatically exclude the possibility that a particular solution is algebraic. (For instance, the linear differential equation $(xy')' = 0$ has not only the algebraic solution 1, but also the transcendental holonomic solution $y(x) = \log(x)$.) However, a recent refinement of Singer's 1979 method can be used to solve *in principle* Stanley's open problem [124, 4(a)]: *given a holonomic power series $y(x)$ by an annihilating linear differential equation and sufficiently many initial terms in its expansion, decide if $y(x)$ is algebraic or transcendental.* Unfortunately, the corresponding algorithm is too slow in practice, because of its high computational complexity[1]. An interesting question is to find efficient alternatives that are able to answer Stanley's question on concrete difficult examples.

An approach that always works is the algorithmic *guess-and-prove* paradigm (see §3.4): one guesses a concrete polynomial witness, and then post-certifies it. This method is very robust and works perfectly well, but it may fail on examples with minimal polynomial much larger than the input differential equation. For instance, in an open question by Zagier [128], the input differential equations have order 4, but the (estimated) algebraic degree of the desired solution is 155520, hence much too large to allow the computation of the minimal polynomial. (Note that the estimate is obtained using seminumerical methods evoked in §3.5).

We aim at designing various pragmatic algorithmic methods for proving algebraicity or transcendence in such difficult cases. First, the algebraic nature of the holonomic function is tested heuristically, using a mixture of numeric and p-adic methods (e.g., monodromy estimates and p-curvature computations). In cases where transcendence is conjectured, the method we will develop is an application of the minimization algorithms in §3.4.1: after finding a minimal-order ODE, an analysis of singularities is sufficient to decide transcendence, at least for interesting subclasses of inputs (e.g., a certain class of generating series of binomial sums). In cases where algebraicity is conjectured, we plan to apply computational strategies inspired by effective differential Galois theory and effective invariant theory, in particular by the recent work [26].

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[1] It relies, among other costly operations, on factoring differential operators, which is known to be a highly expensive procedure, of complexity $(N\mathcal{L})^{O(r^4)}$, where $\mathcal{L}$ is the bitsize of the input operator, r its order, and $N \leq \exp(\mathcal{L} \cdot 2^r)^{2^r}$ [81]. It also relies on deciding whether a non-linear (Ricatti-type) differential equation of order $r - 1$ has an algebraic solution of degree at most $M := (49r)^{r^2}$; this step itself relies on deciding non-emptiness of a constructible set defined by polynomials in M variables (and potentially huge degrees). It also relies on the famously difficult *Abel's problem: given an algebraic function u, decide if $y'/y = u$ has an algebraic solution.*

3.3.2 Algorithmic determination of algebraic values of E-functions

E-functions are holonomic and entire power series subject to some arithmetic conditions; they generalize the exponential function. The class contains most of the holonomic exponential generating functions in combinatorics and many special functions such as the Airy and the Bessel functions. Given an E-function f represented implicitly by a linear differential equation (and enough initial terms), the question is to determine algorithmically the algebraic numbers α such that $f(\alpha)$ is algebraic. A recent article by Adamczewski and Rivoal [21] proves that the problem is decidable. It relies on important works by Siegel [119], Shidlovskii [118], and Beukers [35]. However, the underlying algorithm has no practical applicability. We will obtain an improved version of this algorithm, by accelerating its bottleneck, which consists in computing a linear differential operator of *minimal order* satisfied by f. This will take advantage of the results obtained in §3.4.1. By continuing the line of work opened in [47], the idea is now to exploit the particular structure of differential equations satisfied by E-functions, and to use bounds produced by calculation on the considered equation rather than theoretical bounds such as “multiplicity lemmas". Our previous improvements will make this algorithm practical. We will also address an extension of the theory that also determines cases of algebraic dependency between evaluations of E-functions [77].

3.3.3 Rational solution of Ricatti-like Mahler equation and hypertranscendence

Mahler equations are functional equations that relate a function $f(z)$ with $f(z^p)$, $f(z^{p^2})$, etc., for some integer $p > 0$. The study of Mahler equations is motivated by Mahler’s work in transcendence, as well as by the study of automatic sequences, produced by finite automata (see §3.1.3). From a computer algebra perspective, the basic tasks concerning Mahler equations are poorly understood, compared to differential or recurrence equations.

Roques designed an algorithm for the computation of the Galois group of Mahler equations of order 2 [116]. This group reflects the algebraic relations between the solutions. So its computation is relevant in transcendence theory. Roques’ algorithm relies on deciding the existence of rational solutions to some nonlinear Mahler equations that are analogues of Riccati differential equations. For this task, Roques proposes an algorithm reminiscent of Petkovšek’s algorithm [112], with an exponential arithmetic complexity as it has to iterate through all monic factors of well-identified polynomials. Building on recent progress in the linear case [61], we want to obtain a polynomial-time algorithm for this decidability problem, or at least one that is not exponentially sensitive to the degree of the polynomial coefficients of the equation.

An application of this work will be a new algorithm to decide the differential transcendence of solutions of Mahler equations of order 2, following a criterion given by Dreyfus, Hardouin and Roques (see [69, 116]). This would make it possible to prove new results about some classical Mahler functions and the relations between them. An example will be to reprove and extend the hypertranscendence of the solutions to the Mahler equation satisfied by the generating series of the Stern sequence [67].

3.3.4 Algorithmic determination of algebraic values of Mahler functions

We aim at studying the special values of Mahler functions, going through the search for algebraic values and more generally for algebraic relations between values. We will resume the analysis of the algorithm in [20], to highlight its computational limitations, before optimizing its subtasks. We are thinking in particular of the rationality test, for which an algorithm was given in [28] and another of better complexity has appeared recently [61], and of the search for minimal equations, for which structured linear algebra techniques must allow practical efficiency.

3.3.5 Efficient resolution of functional equations with 1 catalytic variable

In enumerative combinatorics, many classes of objects have generating functions that satisfy functional equations with “catalytic” variables, relating the complete function with the partial functions obtained by specializing the catalytic variables. For equations with a single catalytic variable, either linear or nonlinear, solutions are invariably *algebraic*. This is a consequence of Popescu’s theorem on Artin approximation with nested conditions [115, Thm. 1.4], a deep result in commutative algebra. However, the proof of this *qualitative* result is not constructive. Hence, to go further, towards *quantitative* results, different approaches are needed. Bousquet-Mélou and Jehanne proposed in [49] a method which applies in principle to any equation of the form $P(F(t;x), F_1(t), \dots, F_k(t), t, x) = 0$, where x is a (single) catalytic variable, that admits a unique solution $(F, F_1, \dots, F_k) \in \mathbb{Q}[x][[t]] \times \mathbb{Q}[[t]]^k$. The method is based on a systematic constructive approach, which first derives from the functional equation a (highly structured) algebraic elimination problem over $\mathbb{Q}(t)$ with $3k$ unknowns and $3k$ polynomial equations, whose degree is linear in the degree δ of the input functional equation. The problem is already nontrivial for $k = 1$, but most interesting combinatorial applications require $k > 1$, and current methods are only able to tackle functional equations with small values of k (at most 3) and small total degree δ (at most 4). We will provide unified, systematic and robust algorithms for computing polynomial equations exhibiting the algebraicity of solutions for functional equations with one catalytic variable, building on [49]. The ideal goal is to be able to exploit the geometry and symmetries of the elimination problems arising from the approach in [49]. The final objective is to produce efficient implementations that can be used by combinatorialists in order to solve their functional equations with one catalytic variable *in a click*.

3.3.6 Classification of solutions for functional equations with 2 catalytic variables

When several catalytic variables are involved, Popescu's theorem does not hold anymore. The solutions are not necessarily algebraic anymore, and even holonomy is not guaranteed, even in the linear case.

In the linear case, our the main objective is to fully automatize the resolution of linear equations with two catalytic variables coming from lattice walk questions, when the walk model admits Tutte invariants and decoupling functions. A first nontrivial challenge will be to produce a new computer-assisted proof of algebraicity for the famous Gessel model, different in spirit from the first proof [42]: instead of guess-and-proof, we will be inspired by the recent "human" proofs in [48, 30] relying on Tutte invariants. There are several nontrivial subproblems, both on the mathematical and algorithmic sides. One of them is to determine if a model admits invariants and decoupling functions, and if so, to compute them. A first step in this direction was recently done by Buchacher, Kauers and Pogudin [53], in the simpler case when one looks for *polynomials* instead of rational functions.

In the nonlinear case with two catalytic variables, few results exist, and almost no general theory. These equations occur systematically when counting planar maps equipped with an additional structure, for instance a colouring (or, a spanning tree, a self-avoiding walk, etc.). On this side, the study will be of a more prospective nature. However, we envision the resolution of several challenges. A first objective will be to test various guess-and-prove methods on Tutte's equation [127] satisfied by the generating function of properly q-colored triangulations of the sphere. Any kind of progress on it will be an important success–for instance, proving algebraicity of the solution by a computer-driven approach even for particular values of q such as $q = 2$ and $q = 3$. A second objective will be to automatize the strategy based on Tutte invariants employed by Bernardi and Bousquet-Mélou, and to solve the more general equation (Potts model on planar maps) in an automated fashion. This is interesting already for $q = 2$; in this case, proofs already exist in [29], but they use various ad-hoc tricks. We aim at solving the conjectures in [29] for $q = 3$, concerning the enumeration of properly 3-colored near-cubic maps, by any combination of methods (guess-and-prove, geometric-driven elimination for structured polynomial systems, Tutte invariants).

3.3.7 Deciding integrality of a sequence

Given enough terms of a sequence, it is possible to reconstruct a linear recurrence relation of which the sequence is a solution, if there is one. For example, with the nine numbers 1, 3, 13, 63, 321, 1683, 8989, 48639 and 265729, one can reconstruct the recurrence relation $(n+1)u_n - (6n+9)u_{n+1} + (n+2)u_{n+2} = 0$ for the Delannoy numbers. We would also like to be able to reconstruct the closed form $u_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$, because it reveals arithmetic information absent from the recurrence, such as the integrality of the numbers u_n. The search for a closed form can start by obtaining candidates in a heuristic way, since the summation algorithms make it possible to rigorously prove or disprove *a posteriori* that the reconstructed closed form is indeed correct.

3.3.8 Algorithmic resolution of Padé-approximation problems

Most of the proofs of irrationality of some constant c construct a sequence of rational numbers approximating c with a tight control on the growth of the denominator. Typically, $c = F(1)$ for some holonomic function F, and approximations of F by rational functions may lead to rational approximations of c, by evaluating at 1. Good candidates for approximating F are the *Padé approximants* of F, originating in Hermite's work [87]. But approximations that actually lead to interesting Diophantine results are rare gems. More recently, a general course of action has emerged [36, 123, 78] to deal with the case of multiple zeta values (MZV). It is based on the simultaneous approximation of polylogarithm functions by rational functions. We are looking to automate this approach and to extend its field of application.

We will use computer-assisted symbolic and numerical computations for the construction of a relevant Padé-approximation problem. Then, *the resolution of the problem must be automated*. This is fundamentally a computational problem in a holonomic setting. The natural approach here is *guess-and-prove*: we first guess what could be a closed-form formula for the solution by computing explicitly the solutions for some fixed values of n, then we prove that the guess indeed leads to a solution (which must be unique if the original problem is well-posed). The last step will typically use symbolic integration and Gröbner bases. Similar guess-and-prove approaches in a holonomic setting already gave several Diophantine results [129] but Padé approximation has not been tackled yet in this way.

3.3.9 Software

Our future algorithm for computing a linear differential equation of minimal order satisfied by a given holonomic function will be implemented and made available to users. This may include the application to the determination of algebraic values of E-functions. We will do the same concerning linear Mahler equations of minimal order satisfied by given Mahler functions, and concerning the determination of their algebraic values. Our work on solving equations with catalytic variables has started rather recently, so it is still too early to decide the form that related software should take, but we definitely ambition to provide combinatorialists with an implementation that exhibits the algebraic and/or differential equations they are after.

3.4 Guess-and-prove

Pólya has theorized and popularized a "guess-and-prove" approach to mathematics in remarkable books [114, 113]. It has now became an essential ingredient in experimental mathematics, whose power is highly enhanced when used in conjunction with modern computer algebra algorithms. This paradigm is a key stone in recent spectacular applications in experimental mathematics, such as [42][95, 96]. The first half (the *guessing* part) is based on a "functional interpolation" phase, which consists in recovering equations starting from (truncations of) solutions. The second half (the *proving* part) is based on fast manipulations (e.g., resultants and factorization) with exact algebraic objects (e.g., polynomials and differential operators).

In what follows we mostly focus on the *guessing* phase. It is called *algebraic approximation* [51] or *differential approximation* [93], depending on the type of equations to be reconstructed. For instance, differential approximation is an operation to get an ODE likely to be satisfied by a given approximate series expansion of an unknown function. This kind of reconstruction technique has been used at least since the 1970s by physicists [83, 82, 90], under the name *recurrence relation method*, for investigating critical phenomena and phase transitions in statistical physics. Modern versions are based on subtle algorithms for Hermite–Padé approximants [27]; efficient differential and algebraic guessing procedures are implemented in most computer algebra systems.

In the following subsections, we describe improvements that we will work on.

3.4.1 Univariate guessing

Minimization. A first task is to optimize the search for the minimal-order ODE satisfied by a given holonomic series. Feasibility is already known from the recent [21, §3], but the corresponding algorithm is not efficient in practice, because it relies on pessimistic degree bounds and on pessimistic multiplicity estimates. We will design and implement a much more efficient minimization algorithm, which will combine efficient differential guessing with a dynamic computation of tight degree bounds.

Post-certification. "Multiplicity lemmas" are theorems concluding that an expression representing a formal power series is exactly zero under the weaker assumption that the expression is zero when truncated to some order. In general, the expression is a differential polynomial in a series, but interesting subcases are non-differential polynomials, to test algebraicity, and linear differential expressions, to test holonomicity. In good situations, multiplicity lemmas turn guessing into a proving method or even a decision algorithm. A particularly nice form of a multiplicity lemma is available for polynomial expressions [40, Ch. 7], and a similar result exists for linear ODEs [33]. We will implement such bounds as proving procedures, and we will generalize the approach to other kinds of expressions, e.g., expressions in divided-difference operators that appear in combinatorics, e.g., in map enumeration [49].

Recombination. Generating functions appear in a variety of classes of increasing complexity, in relation to the equations they satisfy. A third subtask relates to the search for an element in a lower complexity class inside the solution set of a higher complexity class. For instance, can a linear or some other combination of non-holonomic series be holonomic? Can a linear combination of holonomic series be algebraic, or even rational? A promising ongoing result, obtained incidentally in the work on Riccati-type solutions for Mahler equations (see §3.3.3), performs a similar guessing by a suitable search for constrained Hermite–Padé approximants after computing the whole module of approximants. But the main expected impact of the approach would be for differential analogues, and we will strive to generalize the approach, taking advantage of the formal analogy between many types of linear operators.

Preparing data. As guessing often requires to first prepare a lot of data, developing fast expansion algorithms for classes of equations is also related to guessing. In this direction, we plan to design a fast algorithm for the high-order expansion of a DD-finite series (i.e., series satisfying linear differential equations with holonomic coefficients). The complexity of the homologue problem for a linear ODE with series coefficients is quasi-linear in the truncation order; that for a linear ODE with polynomial coefficients is just linear. For DD-finite series, we plan to interlace the two approaches without first expanding the series coefficients of the input equation to the wanted order, so as to avoid a large constant and a logarithmic factor.

3.4.2 Multivariate guessing

Multivariate aspects of guessing relate to activities that we plan to develop as a means of strengthening scientific collaborations with colleagues in Paris (PolSys, Sorbonne U.) and Linz (Johannes Kepler University Linz, Austria). How soon the research happens will depend on how interaction with those colleagues evolves.

Trading order for degree. An established technique in the univariate case is known as “trading order for degree”. It is based on the observation that minimal order operators tend to have very high degree, while operators of slightly higher order often have much smaller degrees and are therefore easier to guess. A candidate for the minimal order operator is then obtained as greatest common right divisor of two guessed operators of nonminimal order. We will extend this successful technique to the multivariate case. The desired output in this case is a Gröbner basis of a zero-dimensional annihilating ideal. The coefficients of the Gröbner basis elements are high-degree polynomials, and the idea is, as in the univariate case, not to guess them directly, but to guess ideal elements of smaller total size and to compute the Gröbner basis of them. As Gröbner basis computations can be costly, the alternative operators will clearly already have to be “close” to a Gröbner basis in order for the idea to be beneficial. The questions are: what should close to a Gröbner basis mean, how close should the operators be chosen, how much degree drop can be expected then, and how do the answers to these questions depend on the monomial order?

Exploiting nested structures. In another direction, we plan to exploit the generalized Hankel structure of the matrices that appear when modeling linear recurrence relations guessing through linear algebra. Regarding relations with constant coefficients, this finds applications in polynomial system solving through the `spFGLM` algorithm [72, 73] for finding a lexicographic Gröbner basis. The linear system is block-Hankel with blocks sharing the same structure, and this recursive structure has the same depth as the number of variables. Yet, up to now, only one layer of the structure is handled using fast univariate polynomial arithmetic, then the other ones are dealt with by noting that the matrix has a quasi-Hankel structure and using fast algorithms for this type of matrix [41]. However, the displacement rank of this matrix is not small; hence, not taking into account the full structure of the matrix is suboptimal. This is related to [32] for computing linear recurrence relations with constant coefficients using polynomial arithmetic and [108] for computing multivariate Padé approximants. Analogously, the linear system modeled for guessing linear recurrence relations with polynomial coefficients is highly structured. It is the concatenation of matrices as above, yet these matrices are not independent, as they are all built from the same sequence. Even in the univariate case, the Beckermann–Labahn algorithm is not able to exploit this extra structure in order to be quasi-optimal in the input size. Hence, we would like to investigate how to do so.

In addition to the structure in the modeling, we want to exploit the structure of the sequences that come from applications. For instance, in the enumeration of lattice walks, the nonzero terms often lie in a cone and a lattice, and they are invariant under the action of a finite group. The goal is to take this structure into account in order to build smaller systems for the guessing, and to avoid the generation of more sequence terms than necessary.

3.4.3 Software

We will implement fast algorithms for computing Hermite–Padé approximants of various types [27]. This will include modular integers, integers (via modular reconstruction), simple approximants, and simultaneous approximants. With such a fast, robust implementation at hand, we will also be able to address the guessing of algebraic differential equations (ADE), going beyond the linear case. Our use of state-of-the-art algorithms for computing approximants (including the "superfast" one) will ensure that we outperform earlier implementations such as `Guess` (by Hebisch and Rubey) and `GuessFunc` (by Pantone). We will also develop a variant of trading order for degree for the nonlinear setting. Our implementation will automate the critical selection of derivatives, powers, and coefficient degrees needed to reconstruct an ADE.

3.5 Seminumerical methods in computer algebra

The methods in this research axis deal directly with numbers but, following Knuth [94], they are properly called seminumerical because they lie on the borderline between symbolic and numeric computations. While numerical methods process numerical data and generate further numerical data, our seminumerical methods process exact data, generate high-precision numerical data and reconstruct exact data. In this perspective, the basic unit is not the IEEE-754 floating-point number, but arbitrary precision numbers, typically several thousand decimal places, sometimes more. The crux is not numerical stability, but computational complexity as the number of significant digits goes to infinity. When a number is known at such a high precision, it reveals fundamental structures: rationality, algebraicity, relations with other constants, etc. High-precision computation is a recurring useful tool in the field of experimental mathematics [24]. In some situations, it enables a guess-and-prove approach. In some others, we are unable to step from "guess" to "prove" but overwhelming numerical evidence is enough to shape a conviction. A celebrated example is the experimental discovery of the BBP formula for π [25] (that was proved after its initial guessing). More recently, all the conjectures (some of which became theorems) about multiple zeta values, a hot topic in number theory and mathematical physics, start from high-precision numerical data.

3.5.1 Seminumerical algorithms for linear differential equations

We promote linear differential equations as a data structure to represent and compute with functions (see §3.1). In truth, this data structure represents functions up to finitely many constants. It determines a global behavior but misses the pointwise aspect. Seminumerical methods combine both. They are an important tool for experimental mathematics because they can give strong indications about the nature of a function in very general situations (see §3.3.1).

Factorization. Alexandre Goyer and Raphaël Pagès started a PhD thesis on the factorization of differential operators. It is a fundamental operation for solving linear differential equations, or, at least, elucidate the nature of the solutions. Goyer considers seminumerical methods. They rely on numerical evaluations of the solutions of the differential operators to guess numerically a factorization. High precision makes it possible to reconstruct the factors exactly, and a simple multiplication certifies the computation. Pagès considers a discrete analogue of numerical evaluation: reduction modulo a prime number.

Effective analytic continuation. The main tool for computing high-precision evaluations of functions or integrals is effective analytic continuation of solutions of linear differential equations. It is a form of numerical ODE solver, specialized for linear equations and able to carry out high precision all along the continuation path.

Numerical ODE solvers are a very classical topic in numerical analysis [54], with popular methods, like Runge–Kutta or multistep methods. A much less known family of symbolic-numeric algorithms, that we could call *rigorous Taylor methods*, originates from works of the Chudnovskys' in the 1980s and 1990s [60, 59] and has later been developed by van der Hoeven [88] and Mezzarobba [105, 106]. This family of algorithms only handles linear ODEs with polynomial coefficients, which is precisely the nature of ODEs arising in the context of this document. But contrary to classical methods, they provide very strong guarantees even in difficult situations, especially rigorous error bounds and correct behavior at singular points, all very desirable features in experimental mathematics. Furthermore, they feature a quasi-optimal complexity with respect to precision, meaning that one can compute easily with thousands digits of precision: computing twice as many digits takes roughly twice as much time. This contrasts with fixed-order methods, which cannot reach such precision. For example, to compute 10,000 digits, the classical order four Runge–Kutta method would need typically 10^{2500} steps. This quest for precision is important and crucial in experimental mathematics and theoretical physics [24].

Yet, as advanced as these algorithms may well be, they struggle with the huge ODEs coming from our applications. The reason is easily explained: most algorithms and implementations are designed for small operators and large precision and focuses on a quasilinear complexity with respect to precision. Our situation is quite opposite, with large ODEs and comparatively modest precision. It may be interesting to consider quadratic-time algorithms, with respect to precision, if the complexity with respect to the size of the ODE gets better. This is a really blocking issue that must be addressed to enable new applications. To solve the problem, we will endeavor to provide new software that pays attention to implement algorithms for all regimes of degrees and orders but moderate precision.

3.5.2 Period computation

Periods are numerical integrals that can be computed to high precision with symbolic-numeric integration, even though current algorithms are far from enough to tackle real applications in algebraic geometry, beyond the case of curves. Algorithms for computing periods of curves are mature [66, 109, 107, 63, 52] and have been used, for example, for the computation of the endomorphism ring of genus 2 curves in the LMFDB [64]. Algorithms in higher dimension are only emerging [71, 65, 117]. Their current status does not make them suitable for many applications. Firstly, they are limited in generality. The articles [71, 65] deal with special double coverings of $\mathbb{P}^2$ or $\mathbb{P}^3$, with a low precision, while [117] deals with smooth projective hypersurfaces. In terms of efficiency, we are only able to treat some lucky quartic surfaces (and some very special quintic surfaces or cubic threefolds) for which the underlying ODEs are not too big.

With current methods, we managed to compute the periods of 180 000 quartic surfaces defined by sparse polynomials [99]. This corpus of quartic surfaces was discovered by a random walk. Actually, we are not able to compute (in a reasonable amount of time) the periods of a *given* quartic surface. So we resorted to a random walk guided by ease of computation. This hinders severely the applicability. Yet, this shows the feasibility of transcendental continuation to obtain algebraic invariants that are currently unreachable by any other mean.

The seminumerical algorithms that we develop open perspectives in algebraic geometry. Some integrals with algebraic origin, called periods, convey some interesting algebraic invariants. High-precision computation may unravel them where purely algebraic methods fail [99]. These algebraic invariants are crucial to determine the fine structure of algebraic varieties. We aim at designing algorithms to compute periods efficiently for varieties of general interest, in particular K3 surfaces, quintic surfaces, Calabi–Yau threefolds and cubic fourfolds.

3.5.3 Scattering amplitudes in quantum field theory

In quantum field theory, Feynman integrals appear when computing scattering amplitudes with perturbative methods. In practice, computing Feynman integrals is the most effective way to obtain predictions from a quantum field theory. Precise prediction requires higher-order perturbative terms leading to more complex integrals and daunting computational challenges. For example, [23] reports on the methods used, the difficulties encountered and the limitations met when computing precision calculation for teraelectronvolt collisions in the Large Hadron Collider (LHC).

As far as mathematics is concerned, Feynman integrals are periods. Although this makes the evaluation of Feynman integrals look like *just* a special case of symbolic-numeric integration, it would be naive to pretend that our methods apply without effort: it is clear that the computations are so challenging that only specialized methods may succeed. Current methods include sector decomposition [122] (where the integration domain is decomposed in smaller pieces on which traditional numerical integral algorithms perform well) and the use of differential equations [85] in a similar fashion to what we propose here, namely the symbolic computation of integrals with a parameter combined with numerical ODE solving. In the longer term, we expect that an efficient toolbox to deal with holonomic ideals would improve computations with Feynman integrals. It is however too early to say.

In the short term, the experimental mathematics toolbox that we want to develop may be useful to understand the geometry underlying some Feynman integrals. The typical outcome is simple analytic formulas [38, 37] allowing for fast and precise computations. In this context, identifying key algebraic invariants before engaging further mathematical thinking is crucial. For example, a key fact in the analysis of a three-loop graph by [37] is the generic member of some family of K3 surfaces having Picard rank 19. For other graphs appear cubic fourfolds which we cannot investigate numerically at the moment. An expected outcome of the previously exposed objectives is the computation of the periods of such varieties. This is a first step towards a more systematic development of this interface with high-energy physics.

3.5.4 Software

Solid software foundations for effective analytic continuation (see §3.5.1) will be important for the other tasks in this section. We use currently the part of the package ore_algebra developed by Marc Mezzarobba, but it is a bottleneck for several algorithms. The plan for the software development (improvement of ore_algebra, or whole new package) is not fixed yet: it depends on the nature of the algorithmic ideas that will emerge.

4 Application domains

As already expressed in §2.3, our natural application domains are:

- Combinatorics,
- Probability theory,
- Number theory,
- Algebraic geometry,
- Statistical physics,
- Quantum mechanics.

5 Highlights of the year

5.1 Awards

- Alexandre Guillemot was awarded the Distinguished Software Presentation Award at the *50th International Symposium on Symbolic and Algebraic Computation* (ISSAC 2025, Guanajuato), for his presentation “Certified Algebraic Path Tracking with Algpath” on the Rust package Algpath (see §6.1.1).

6 Latest software developments, platforms, open data

6.1 Latest software developments

6.1.1 Algpath

Keywords: Interval arithmetic, Polynomial equations

Functional Description: Algpath is a Rust package for the rigorous computation of the continuation of a regular zero of a parametrized polynomial system as the parameter varies.

URL: `https://gitlab.inria.fr/numag/algpath`

Contact: Alexandre Guillemot

Participants: Alexandre Guillemot, Pierre Lairez

7 New results

Participants: Hadrien Brochet, Frédéric Chyzak, Philippe Dumas, Guy Fayolle, Claudia Fevola, Alexandre Goyer, Alexandre Guillemot, Pierre Lairez, Rafael Mohr.

7.1 Conway's cosmological theorem and automata theory

John Conway proved that every audioactive sequence (a.k.a. *look-and-say*) decays into a compound of 94 elements, a statement he termed the *cosmological theorem*. The underlying audioactive process can be modeled by a finite-state machine, mapping one sequence of integers to another. Leveraging automata theory, Pierre Lairez and Aleksandr Storozhenko propose a new proof of Conway's theorem based on a few simple machines, using a computer to compose and minimize them [5]. The article was published in 2025 in *The American Mathematical Monthly*.

7.2 Wronski pairs of honeycomb curves

In [1], Laura Casabella (MPI-MiS, Leipzig), Michael Joswig (TU Berlin), and Rafael Mohr studied certain generic systems of real polynomial equations associated with triangulations of convex polytopes and investigated their number of real solutions. The main focus was set on pairs of plane algebraic curves which form a so-called Wronski system. The computational tasks arising in the analysis of such Wronski pairs lead the authors to the frontiers of current computer algebra algorithms and their implementations, both via Gröbner bases and numerical algebraic geometry.

7.3 A syzygial method for equidimensional decomposition

Based on a theorem by Vasconcelos, Rafael Mohr gave in [6] an algorithm for equidimensional decomposition of algebraic sets using syzygy computations via Gröbner bases. His algorithm avoids the use of elimination, homological algebra and processing the input equations one-by-one present in previous algorithms. The practical interest of this algorithm was demonstrated experimentally compared to the state of the art.

7.4 On the computation of Newton polytopes of eliminants

In [8], for systems of polynomial equations, Yulia Mukhina (LIX, École Polytechnique) and Rafael Mohr studied the problem of computing the Newton polytope of their eliminants. As was shown by Esterov and Khovanskii, such Newton polytopes are mixed fiber polytopes of the Newton polytopes of the input equations. The authors used these results in combination with mixed subdivisions to design an algorithm computing these special polytopes. The increase in practical performance of this algorithm compared to existing methods using tropical geometry was demonstrated experimentally and the differences that lead to this increase in performance was discussed. The authors also demonstrated an application of their work to differential elimination.

7.5 First-order factors of linear Mahler operators

In [2], Frédéric Chyzak and Philippe Dumas, together with Thomas Dreyfus (Université de Bourgogne) and Marc Mezzarobba (LIX), developed and compared two algorithms for computing first-order right-hand factors in the ring of linear Mahler operators $\ell_r M^r + \cdots + \ell_1 M + \ell_0$ where $\ell_0, \ldots, \ell_r$ are polynomials in x and $Mx = x^b M$ for some integer $b \geq 2$. In other words, they gave algorithms for finding all formal infinite product solutions of linear functional equations $\ell_r(x) f(x^{b^r}) + \cdots + \ell_1(x) f(x^b) + \ell_0(x) f(x) = 0$.

The first of their algorithms is adapted from Petkovšek's classical algorithm for the analogous problem in the case of linear recurrences. The second one proceeds by computing a basis of generalized power series solutions of the functional equation and by using Hermite–Padé approximants to detect those linear combinations of the solutions that correspond to first-order factors.

In their article, which was published in 2025, they presented implementations of both algorithms and discussed their use in combination with criteria from the literature to prove the differential transcendence of power series solutions of Mahler equations.

7.6 Differential equations satisfied by generating functions of 5-, 6-, and 7-regular labelled graphs: a reduction-based approach

By a classic result of Gessel, the exponential generating functions for k-regular graphs are D-finite. Using Gröbner bases in Weyl algebras, Frédéric Chyzak and Marni Mishna (Simon Fraser University) computed the linear differential equations satisfied by the generating function for 5-, 6-, and 7- regular graphs [14]. Their method is sufficiently robust to consider variants such as graphs with multiple edges, loops, and graphs whose degrees are limited to fixed sets of values. The article was accepted in 2025.

7.7 Faster multivariate integration in D-modules

Hadrien Brochet, Frédéric Chyzak, and Pierre Lairez presented a new algorithm for solving the reduction problem in the context of holonomic integrals, which in turn provides an approach to integration with parameters. Their method [12] extends the Griffiths-–Dwork reduction technique to holonomic systems and Hadrien Brochet implementeded it in Julia. While not yet outperforming creative telescoping in D-finite cases, it enhances computational capabilities within the holonomic framework. As an application, they derived a previously unattainable differential equation for the generating series of 8-regular graphs, thus prolonging the results obtained by Frédéric Chyzak and Marni Mishna in [14].

7.8 Diagonals of permutahedra and associahedra

Alin Bostan (in the team until last year) and Frédéric Chyzak, together with a few other colleagues, presented enumeration formulas for the faces of cellular diagonals of the permutahedra and associahedra. For the former, they used Zaslavsky's theory to count the faces of the hyperplane arrangement obtained as the union of ℓ generically translated copies of the braid arrangement. This yields in particular nice formulas for the number of regions and bounded regions in terms of exponentials of generating functions of Fuss—Catalan numbers. For the latter, they used analytic or bijective methods to enumerate Tamari intervals weighted by certain binomial coefficients, leading to a surprisingly simple product formula. An article was published in FPSAC 2025 [7].

7.9 Single-exponential bounds for diagonals of D-finite power series

D-finite power series appear ubiquitously in combinatorics, number theory, and mathematical physics. They satisfy systems of linear partial differential equations whose solution spaces are finite-dimensional, which makes them enjoy a lot of nice properties. After attempts by others in the 1980s, Lipshitz was the first to prove that the class they form in the multivariate case is closed under the operation of diagonal. In particular, an earlier work by Gessel had addressed the D-finiteness of the diagonals of multivariate rational power series. In [13], Frédéric Chyzak, his long-term colleague Shaoshi Chen from the Chinese Adacemy of Sciences (Beijing), a joint PhD student Pingchuan Ma who visited six months in 2024, and another student Chaochao Zhu gave another proof of Gessel's result that fixes a gap in his original proof, while extending it to the full class of D-finite power series. They also provided a single exponential bound on the degree and order of the defining differential equation satisfied by the diagonal of a D-finite power series in terms of the degree and order of the input differential system.

7.10 Algebraic and positive geometry of the universe: from particles to galaxies

In recent years, the intersection of algebra, geometry, and combinatorics with particle physics and cosmology has led to significant advances. Central to this progress is the twofold formulation of the study of particle interactions and observables in the universe: on the one hand, Feynman's approach reduces to the study of intricate integrals; on the other hand, one encounters the study of positive geometries. In [4], Claudia Fevola and Anna-Laura Sattelberger introduced key developments, mathematical tools, and the connections that drive progress at the frontier between algebraic geometry, the theory of D-modules, combinatorics, and physics. All these threads contribute to shaping the flourishing field of positive geometry, which aims to establish a unifying mathematical language for describing phenomena in cosmology and particle physics.

7.11 Tropical KP theory on banana curves

The Kadomtsev-Petviashvili (KP) equation is the cornerstone of integrable systems, whose solutions reflect deep connections in algebraic geometry. Banana curves are reducible rational curves obtained as a degeneration of hyperelliptic curves. In [11], Claudia Fevola together with Simonetta Abenda, Türkü Özlüm Çelik, and Yelena Mandelshtam related the family of KP multi-solitons arising from banana curves together with non-special divisors of fixed degree to the combinatorics of the tropical theta divisor of the curve. They described the Voronoi and Delaunay polytopes and show that the latter are combinatorially equivalent to uniform matroid polytopes. As a consequence, the combinatorics of the tropical theta divisor canonically encodes the matroid and Grassmannian structures underlying the associated KP multi-soliton solutions. The authors defined the Hirota variety of a banana graph, which parametrizes all tau functions arising from such a graph. Starting from the matroid arising from Delaunay polytopes and the periods in the tropical limit, they constructed an explicit parametrization of this variety which realizes the tau function as a multi-soliton. Their framework specializes naturally to real and positive settings.

7.12 Symbolic-numeric algorithms in differential algebra

Alexandre Goyer defended his PhD thesis in 2025 [10]. His work focused on the design of practically efficient algorithms for manipulating and factorizing linear differential equations with rational function coefficients. Factorization aims at decomposing an equation into lower-order operators in order to better understand solution spaces and simplify computations. After reviewing early methods, from Beke's algorithm to its 20th-century improvements, the thesis highlights their major practical limitations. It then discusses the more efficient eigenring and local-to-global approaches developed in the 1990s, which form the basis of current state-of-the-art algorithms but remain incomplete. The work follows the symbolic-numeric direction initiated by van der Hoeven, exploiting numerical approximations obtained via analytic continuation of solutions. It relies in particular on high-precision, rigorous numerical tools developed by Mezzarobba. The manuscript introduces the algebraic framework of noncommutative differential operators and essential analytic and Galois-theoretic concepts. Building on these foundations, Alexandre Goyer presented a new symbolic-numeric factorization algorithm combining and extending existing methods, while reducing the reliance on costly monodromy computations. The algorithm is implemented in SageMath and tested on a wide range of examples from several application domains. Experiments showed that it often outperforms Maple's DEtools module and suggested promising extensions to Loewy decompositions.

7.13 Efficient algorithms for creative telescoping using reductions

The need for computing multiple integrals with parameters in several areas of mathematics has led to the development of many algorithms in the field of symbolic integration. Yet, some applications are currently hindered by the lack of generality or efficiency of state-of-the-art algorithms. This motivated the study of integrals in the context of D-modules. The goal of Hadrien Brochet's PhD thesis was to introduce new ideas to unlock this situation: designing and implementing a signature-based algorithms for computing Gröbner bases in Weyl algebras with nice practical behavior; relaxing the emphasis on minimality of current algorithms have by using a more liberal termination criterion only based on holonomy; using generic deformations and restriction for building-block algorithms. Hadrien Brochet defended his PhD thesis in 2025 [9].

7.14 Certified algebraic path tracking with Algpath

Algpath is a certified homotopy continuation software. In [17], Alexandre Guillemot upgraded the previous fixed-precision Rust implementation by incorporating mixed, adaptive precision with minimal overhead. This allowed him to tackle problems on which the initial implementation failed due to the inability to increase precision, and where uncertified methods may have fail or path jump.

7.15 Stability and renormalization of Jackson networks endowed with a finite pool of greedy mobile servers

A tandem of two queues sharing a pool of servers, where users take the time to switch to the second queue, is used to model a typical pathway through an emergency department (ED), where patients undergo two consultations separated by diagnostic tests. In [15], Ch. Fricker (Inria Paris) and Guy Fayolle obtained explicit conditions for ergodicity and transience, which they proved via Foster's criterion, by using a linear Lyapunov function. They extended this result to a Jackson network, with the key difference that the nodes share a pool of servers, with a non-idling service policy. Further, the delay times for customers to move from one node to another must be taken into account. This covers some of the main features of models for emergency departments, namely priorities (triage) between patients. In the case of the tandem queue, scaling the arrival rate and the number of servers by N yields a renormalized process that converges to the solution of an ordinary differential equation (ODE) with boundary conditions. In the case of stability, the nature of this ODE as $t \to \infty$ was also discussed.

7.16 Time-scaling of stop-and-go waves in car-following models

Waves, known as *stop-and-go waves* or *phantom jams*, can appear spontaneously in dense traffic. This causes a situation where drivers are faced with consecutive phases of acceleration and braking. Although waves are well understood in the setting of macroscopic models, the results for car-following models are not so numerous. Starting from the linearization of these models, and assuming string instability, Guy Fayolle and J.-M. Lasgouttes (Inria Paris) [16] gave asymptotic estimates of the velocity and shape of these waves. Their result relies on the well-known saddle-point method in order to describe the trajectory of a vehicle caught in such a wave. Numerical experiments have shown that this method yields remarkably good estimates of the linearized model, even with only 5 vehicles, as well as a good estimate of the wave velocity.

7.17 Thermodynamical limits for models of car-sharing systems: the Autolib' example

Ch. Fricker (Inria Paris) and Guy Fayolle analyzed in [3] various mean-field equations obtained for models involving a large station-based car-sharing system in France called Autolib'. Their focus is mainly on a version without capacity constraints, where users reserve a parking space when they take a car. The model is carried out in thermodynamical limit, that is when the number N of stations and the fleet size M_N tend to infinity with $U = \lim_{N\to\infty} M_N/N$. This limit is described by Kolmogorov's equations of a two-dimensional time-inhomogeneous Markov process depicting the numbers of reservations and cars at a station. It satisfies a non-linear differential system having a unique solution, which converges, as $t \to \infty$, exponentially fast towards an equilibrium point, which corresponds to the stationary distribution of a two-queue tandem (reservations, cars), that is always ergodic. The intensity factor of each queue has an explicit form obtained from an intrinsic mass conservation relationship. Two related models with capacity constraints were also presented: the simplest one with no reservation leads to a one-dimensional problem; the second one corresponds to our first model with finite total capacity K [3].

8 Partnerships and cooperations

Participants: Frédéric Chyzak, Claudia Fevola.

8.1 International research visitors

8.1.1 Visits of international scientists

Other international visits to the team

Shaoshi Chen

Status researcher

Institution of origin: Chinese Academy of Sciences (Beijing)

Country: China

Dates: March 31 to April 4 + December 1 to December 10.

Context of the visit: research collaboration with Frédéric Chyzak + jury member of Hadrien Brochet's PhD defense

Mobility program/type of mobility: research stay

Chiara Meroni

Status researcher

Institution of origin: ETH Institute for Theoretical Studies

Country: Switzerland

Dates: March 31 to April 2.

Context of the visit: speaker at the MATHEXP seminar

Mobility program/type of mobility: research stay

Simon Telen

Status researche group leader

Institution of origin: Max Planck Institute for Mathematics in the Sciences (Leipzig)

Country: Germany

Dates: April 28 to May 2.

Context of the visit: research discussion with Claudia Fevola and speaker at the MATHEXP seminar

Mobility program/type of mobility: research stay

Christoph Koutschan

Status researcher

Institution of origin: RICAM (Linz)

Country: Austria

Dates: December 3 to December 10.

Context of the visit: research collaboration with Frédéric Chyzak + jury member of Hadrien Brochet's PhD defense

Mobility program/type of mobility: research stay

8.1.2 Visits to international teams

Research stays abroad

Frédéric Chyzak

Visited institution: Chinese Academy of Sciences (Beijing)

Country: China

Dates: May 30 to June 6

Context of the visit: prolonged his stay in Beijing after the conference DART for a collaboration with Shaoshi Chen

Mobility program/type of mobility: research stay

Claudia Fevola

Visited institution: Max Planck Institute of Molecular Cell Biology and Genetics, (Dresden)

Country: Germany

Dates: March 24 to 28 and June 30 to July 4

Context of the visit: research visit for a collaboration with Simonetta Abenda, Türkü Ôzlüm Çelik (host), and Yelena Mandelshtam

Mobility program/type of mobility: research stay

Claudia Fevola

Visited institution: Laboratoire d'Annecy-le-Vieux de Physique Théorique (LAPTh, Annecy)

Country: France

Dates: February 17 to 21

Context of the visit: Scientific discussions with Piotr Tourkine and seminar talk

Mobility program/type of mobility: Interdisciplinary secondment funded by the Postdoctoral fellowship from MathInGreaterParis

Claudia Fevola

Visited institution: University of Amsterdam

Country: Netherlands

Dates: October 27 to November 14

Context of the visit: Scientific discussions with Daniel Baumann and attendance in the workshop Cosmology meets Non-linear Algebra

Mobility program/type of mobility: research stay

8.2 European initiatives

8.2.1 Horizon Europe

- *ERC Starting Grant "10000 DIGITS".* This project led by Pierre Lairez spans for five years starting from April 2022. It funds three PhD theses and three 2-year post-doctoral positions. Its goal is to develop algorithms and software to compute with high precision integrals with a geometric origin, especially periods of algebraic varieties, and to tackle applications in diophantine approximation, quantum field theory, and optimization.

- *Postdoctoral fellowship from MathInGreaterParis.* Claudia Fevola obtained a two-year postdoctoral fellowship hosted by MATHEXP and funded by the MathInGreaterParis Fellowship Programme, cofunded by Marie Sklodowska-Curie Actions H2020-MSCA-COFUND-2020. She obtained a prolongation of her fellowship by a few months at the end of 2025.

9 Dissemination

9.1 Promoting scientific activities

9.1.1 Scientific events: organisation

General chair, scientific chair

- Frédéric Chyzak is part of the scientific committee of the RT EFI ("Functional Equations and Interactions", successor of GDR EFI) dependent on the mathematical institute (INSMI) of CNRS. The goal of this RT (thematic network) is to bring together various research communities in France working on functional equations in fields of computer science and mathematics. The RT is really a "transversal axis", with activities in three RT of INSMI: Algèbre, Géométrie Algébrique et Singularités, and Théorie des Nombres.

Member of the organizing committees

- Ricardo Buring was part of the organizing committee of the *2025 FLINT development workshop in Palaiseau*.

- Claudia Fevola was part of the organizing committee of the 2025 edition of the conference *MathemAmplitudes: Co-homology and Combinatorics of GKZ, Euler-Mellin-Feynman Integrals and Scattering Amplitudes*.

- Pierre Lairez was part of the organizing committee of the 2025 edition of the conference *Journées nationales de calcul formel* (JNCF).

- Rafael Mohr was part of the organizing team of a mini-symposium at the 2025 edition of the conference *SIAM Conference on Applied Algebraic Geometry* (AG25).

9.1.2 Scientific events: selection

Reviewer

- Frédéric Chyzak and Rafael Mohr have served as reviewers for the conference *Computer Algebra in Scientific Computing* (CASC).

9.1.3 Journal

Member of the editorial boards

- Frédéric Chyzak is on the editorial board of the *Journal of Systems Science and Complexity* (JSSC).
- Guy Fayolle is associate editor of the journal *Markov Processes and Related Fields* (MPRF).
- Pierre Lairez is on the editorial board of the *SIAM Journal in applied algebraic geometry* (SIAGA).

Reviewer - reviewing activities

- Frédéric Chyzak has been a reviewer for the international journals: *Combinatorial Theory*, *European Journal of Mathematics*, *Journal of Symbolic Computation*, and *Random Structures & Algorithms*.
- Guy Fayolle has been a reviewer for the international journals: *Electronic Communications in Probability*; *Journal of Statistical Physics*; *Markov Processes and Related Fields*; *Transactions of the American Mathematical Society*.
- Claudia Fevola has been a reviewer for the international journals: *SIAM Journal on Applied Algebra and Geometry*, and *The Journal of the Society for the Foundations of Computational Mathematics*.
- Rafael Mohr has been a reviewer for the international journals: *Journal of Symbolic Computation* and *Mathematics in Computer Science*.

9.1.4 Invited talks

- Ricardo Buring was invited speaker at:
  - the *Journées Nationales de Calcul Formel 2025* (JNCF 2025, Marseille),
  - the *XXIX International Conference on Integrable Systems and Quantum Symmetries* (ISQS29, Prague).
- Frédéric Chyzak was invited speaker at:
  - the conference *Differential Algebra and Related Topics* (DART XIII, Beijing),
  - a one-day workshop in Beijing,
  - the *30th Conference on Applications of Computer Algebra* (ACA 2025, Heraklion),
  - the *ODELIX Thematic Fall Program 2025* (Palaiseau).
- Guy Fayolle was invited speaker at the conference *Un quart de siècle pour un quart de plan*, which was organized to celebrate his achievements in the field of random walks.
- Claudia Fevola was invited speaker at:
  - the conference *Real Sociedad Matemática Española's 7th Congress of Young Researchers* (RSME2025, Bilbao),
  - the weekly seminar at the *Laboratoire d'Annecy-le-Vieux de Physique Théorique* (LAPTh, Annecy),
  - the workshop *Recent Advances in Classical and Quantum Integrability* (Leeds),
  - the workshop *Computations in Algebraic Geometry: Complex, Real, and Tropical* (Zürich),

  - the workshop *Singularities, D-modules, and Connections to Physics* (Banff),
  - the *Dublin Mathematics Colloquium, Geometry Seminar* (Dublin),
  - the online seminar *Algebraic and combinatorial perspectives in the mathematical sciences*.
- Alexandre Guillemot was invited speaker at:
  - the *Journées Nationales de Calcul Formel 2025* (JNCF 2025, Marseille),
  - the *Algèbre et Géométrie* team seminar at the *Laboratoire de Mathématiques de Versailles* (Versailles),
  - the *SIAM Conference on Applied Algebraic Geometry* (SIAM AG25, Madison),
  - the *30th Conference on Applications of Computer Algebra* (ACA 2025, Heraklion),
  - the *Pascaline* team seminar at the *Laboratoire de l'Informatique du Parallélisme* (Lyon),
  - the *Journées de Géométrie Algorithmique* (JGA 2025, Roscoff),
  - the *Ouragan* team seminar at the *Institut de Mathématiques de Jussieu – Paris rive gauche* (Paris).
- Pierre Lairez was invited speaker at:
  - the *Mathemamplitudes* conference (Mainz, Germany),
  - the *Séminaire Darboux* (LPTHE, Paris).
- Rafael Mohr was invited speaker at:
  - the weekly *ECO Seminar* at the Université de Montpellier.
- Théo Ternier was invited speaker at:
  - the *ODELIX Thematic Fall Program 2025* (Palaiseau).

9.1.5 Research administration

- Frédéric Chyzak is a member of the commission of users of computer resources (CUMI) at the Inria Saclay Center.
- Frédéric Chyzak leads the mentoring commission of the Inria Saclay Center. A new campaign has started in 2025, involving a dozen of pairs mentor/mentoree. He is also a mentor in the mentoring program.
- Since 2023, Frédéric Chyzak is an elected member of the evaluation commission (CE) of Inria. In 2025, he served in several national juries and commissions (promotion, DR2 recruitement, C3 bonus).
- Guy Fayolle is scientific advisor and associate researcher at the *Centre for Robotics* (Mines Paris PSL).
- Guy Fayolle is a member of the working group *WG 7.3: Computer System Modeling* of the *International Federation for Information Processing* (IFIP).
- Pierre Lairez is elected substitute member in the *comité de centre* of the Inria Saclay research center.

9.2 Teaching - Supervision - Juries - Educational and pedagogical outreach

- **Bachelor**:
  - Ricardo Buring, *Object-oriented Programming in C++*, Bachelor Polytechnique, France.
  - Alexandre Guillemot, *Computer Programming (CSC 1S002 EP)*, TD, 24h, Bachelor Polytechnique, France.
- **Licence 1**:
  - Théo Ternier, *Algebra and Geometry*, TD, 24h, Université Paris-Saclay, France.

- **Licence 2**:
  - Théo Ternier, *Differential equations*, TD, 16h, Université Paris-Saclay, France.
- **Master**:
  - Alexandre Guillemot, *Les bases de la programmation et de l'algorithmique (CSC 41011 EP)*, TD, 40h, M1, École polytechnique, France.
  - Pierre Lairez, *Les bases de la programmation et de l'algorithmique (CSC 41011 EP)*, TD, 40h (groups 5 and 10), tutoring, 10h, École polytechnique, France.
  - Pierre Lairez, *Introduction à l'informatique (INF361)*, TD, 40h (groups 4 and 12), École polytechnique, France.

9.2.1 Supervision

- **Master interships**:
  - Claudia Fevola supervised the Master (M2) thesis of Jaali Mazzaggio on "Euler Stratifications of Families of Quadric Hypersurfaces from Particle Physics: 1-Loop Diagrams".
  - Pierre Lairez and Rafael Mohr supervised the Master (M2) thesis of Théo Ternier on "An efficient data structure for monomial ideals and its application to signature Gröbner basis computation".
- **PhD theses**:
  - Frédéric Chyzak co-supervised together with Marc Mezzarobba (CNRS, LIX) the PhD thesis of Alexandre Goyer on "Symbolic-numeric algorithms in differential algebra". The defense took place in 2025.
  - Frédéric Chyzak and Pierre Lairez co-supervised the PhD thesis of Hadrien Brochet on "Algorithms for D-modules". The defense took place in 2025.
  - Frédéric Chyzak and Pierre Lairez co-supervise the PhD thesis of Théo Ternier on "Efficient algorithms for signature Gröbner bases and D-modules".
  - Pierre Lairez supervises the PhD thesis of Alexandre Guillemot on "Effective topology of complex algebraic varieties".

9.2.2 Juries

- Frédéric Chyzak has served as a reviewer and president of the jury for the PhD defense of Lucas Legrand (Université de Limoges), *Gröbner bases over polyhedral algebras*, Sept. 16, 2025.
- Frédéric Chyzak has served as president of the jury for the PhD defense of Camille Pinto (Sorbonne université), *Elimination theory for linear integro-differential systems*, Oct. 23, 2025.
- Frédéric Chyzak has served as president of the jury for the PhD defense of Maxime Bridoux (Université de Rennes), *Inférence et exploitation de conditions nécessaires pour l'existence de polynomes de Darboux polynomials*, Nov. 28, 2025.
- Guy Fayolle was examiner in the jury for Sandro Franceschi's HDR defense (Institut Polytechnique de Paris), *Reflected Stochastic processes in cones*, Dec. 11, 2025.
- Pierre Lairez was examiner in the jury for the PhD defense of Quentin Canu (École polytechnique), Dec. 2025.

10 Scientific production

10.1 Publications of the year

International journals

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[2] F. Chyzak, T. Dreyfus, P. Dumas and M. Mezzarobba. 'First-order factors of linear Mahler operators'. In: *Journal of Symbolic Computation* (2025). DOI: `10.1016/j.jsc.2025.102424`. URL: `https://inria.hal.science/hal-04502052` (cit. on p. 22).

[3] G. Fayolle and C. Fricker. 'Thermodynamical limits for models of car-sharing systems: the Autolib' example'. In: *Markov Processes And Related Fields* 31.1 (2025), pp. 55–78. URL: `https://inria.hal.science/hal-04855046` (cit. on pp. 24, 25).

[4] C. Fevola and A.-L. Sattelberger. 'Algebraic and Positive Geometry of the Universe: from Particles to Galaxies'. In: *Notices of the American Mathematical Society* 72.8 (1st Sept. 2025). DOI: `10.1090/noti3220`. URL: `https://hal.science/hal-05295613` (cit. on p. 23).

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International peer-reviewed conferences

[7] A. Bostan, F. Chyzak, B. Delcroix-Oger, G. Laplante-Anfossi, V. Pilaud and K. Stoeckl. 'Diagonals of Permutahedra and Associahedra'. In: *Proceedings of the 37th International Conference on "Formal Power Series and Algebraic Combinatorics", July 21 - 25, 2025, Hokkaido University, Sapporo, Japan*. 37th International Conference on "Formal Power Series and Algebraic Combinatorics. Vol. 93B. Séminaire Lotharingien de Combinatoire. Sapporo, Japan, 1st Apr. 2025, 12, Art. 136. URL: `https://inria.hal.science/hal-05468595` (cit. on p. 22).

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Doctoral dissertations and habilitation theses

[9] H. Brochet. 'Efficient Algorithms for Creative Telescoping using Reductions'. Université Paris - Saclay, 5th Dec. 2025. URL: `https://hal.science/tel-05474780` (cit. on p. 24).

[10] A. Goyer. 'Symbolic-numeric algorithms in differential algebra'. Université Paris-Saclay, 31st Oct. 2025. URL: `https://theses.hal.science/tel-05473922` (cit. on p. 23).

Reports & preprints

[11] S. Abenda, T. Özlüm Çelik, C. Fevola and Y. Mandelshtam. *Tropical KP Theory on Banana Curves*. 15th Dec. 2025. URL: `https://hal.science/hal-05419464` (cit. on p. 23).

[12] H. Brochet, F. Chyzak and P. Lairez. *Faster multivariate integration in D-modules*. 17th Apr. 2025. URL: `https://hal.science/hal-05289785` (cit. on p. 22).

[13] S. Chen, F. Chyzak, P. Ma and C. Zhu. *Single-exponential bounds for diagonals of D-finite power series*. 20th June 2025. URL: `https://inria.hal.science/hal-05132552` (cit. on p. 23).

[14] F. Chyzak and M. Mishna. *Differential equations satisfied by generating functions of 5-, 6-, and 7-regular labelled graphs: a reduction-based approach*. 2025. URL: https://inria.hal.science/hal-04604501 (cit. on p. 22).

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[16] G. Fayolle and J.-M. Lasgouttes. *Stop-and-Go Waves in Car-Following Models*. INRIA, 2nd Dec. 2025, p. 31. URL: https://inria.hal.science/hal-04854057 (cit. on p. 24).

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10.2 Cited publications

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